Differential Calculus of 3D Orientations

Arun Das

05/15/2017

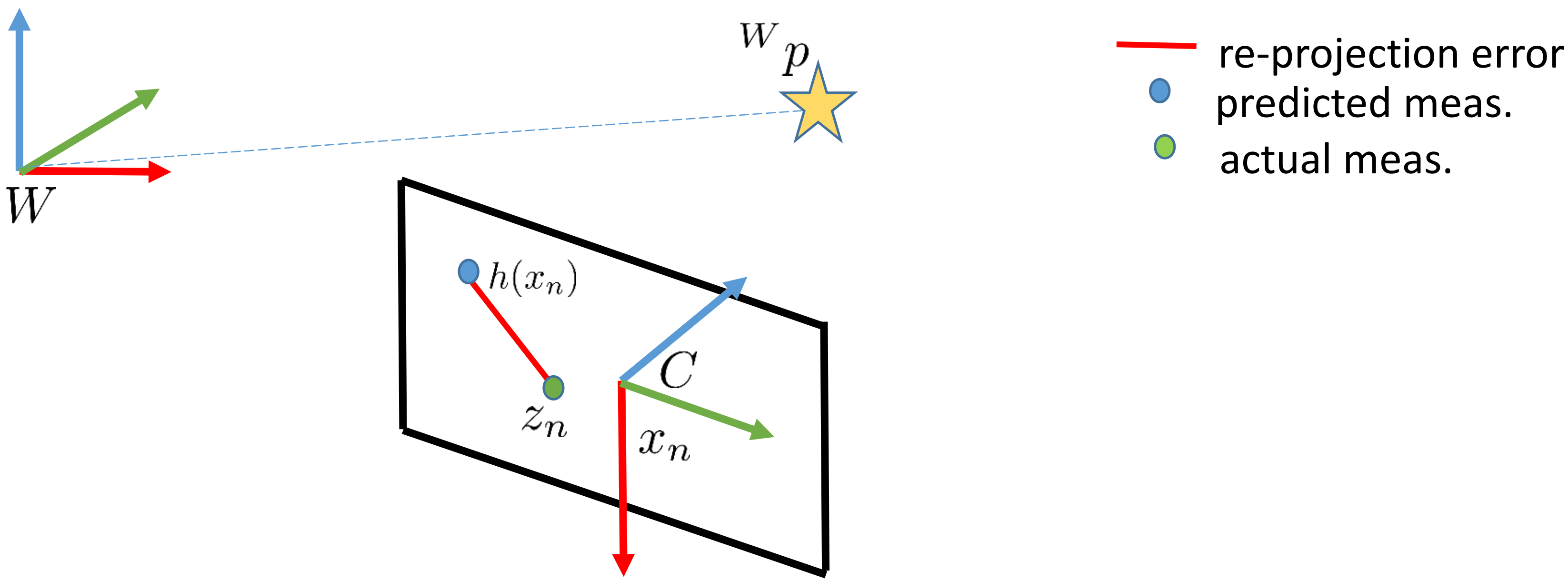
- Quick Review of Lie Groups
- Box-plus and Box-minus
- Derivatives of Rotations (**A Primer on the Differential Calculus of 3D Orientations, Bloesch et al.)**
  - Derivations on the board
- Uncertainty of rotations and rotated vectors (**State Estimation for Robotics, Barfoot**)

$$\mathbf{e}_k(\breve{\mathbf{x}}_k + \boldsymbol{\Delta}\mathbf{x}_k) = \mathbf{e}_k(\breve{\mathbf{x}} + \boldsymbol{\Delta}\mathbf{x}) \simeq \mathbf{e}_k + \mathbf{J}_k \boldsymbol{\Delta}\mathbf{x}.$$

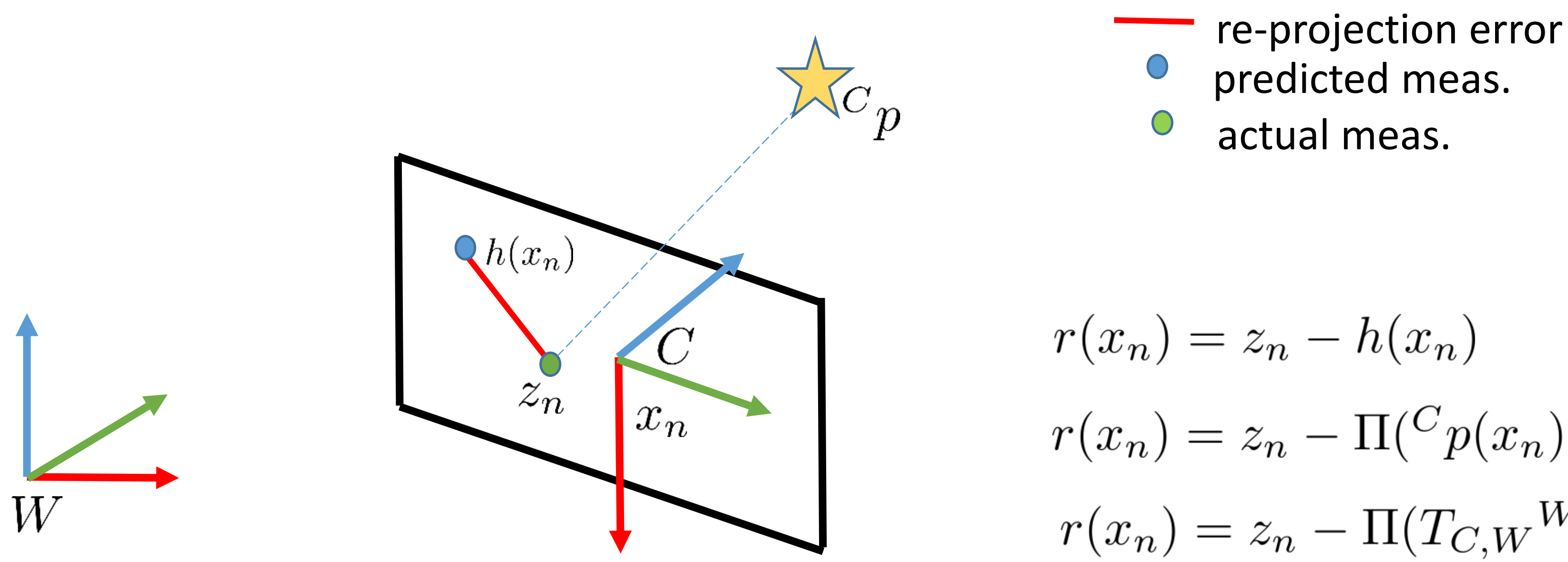
Where J is the **Jacobian of the residual term**

$$\mathbf{H}\,\boldsymbol{\Delta}\mathbf{x}^* = -\mathbf{b}.$$

Important to be able to get analytical Jacobians for our residual terms!



How do we do **re-projection?**



$$r(x_n) = z_n - h(x_n)$$

$$r(x_n) = z_n - \Pi(^C p(x_n))$$

$$r(x_n) = z_n - \Pi(T_{C,W} {}^W p)$$

What is $\frac{\partial r(x_n)}{\partial x_n}$?

Remember that x_n is a **pose**!

Need to use **chain rule**!

Composition: $(\Phi_{\mathcal{C}\mathcal{B}} \circ \Phi_{\mathcal{B}\mathcal{A}})({}_{\mathcal{A}}\boldsymbol{r}) \triangleq \Phi_{\mathcal{C}\mathcal{B}}(\Phi_{\mathcal{B}\mathcal{A}}({}_{\mathcal{A}}\boldsymbol{r}))$

Identity: $\Phi_I \circ \Phi_{\mathcal{B}\mathcal{A}} = \Phi_{\mathcal{B}\mathcal{A}} \circ \Phi_I = \Phi_{\mathcal{B}\mathcal{A}}$

Inverse: $\Phi_{\mathcal{B}\mathcal{A}}^{-1} \circ \Phi_{\mathcal{B}\mathcal{A}} = \Phi_{\mathcal{B}\mathcal{A}} \circ \Phi_{\mathcal{B}\mathcal{A}}^{-1} = \Phi_I$

Lie Group SO(3)
State Vector

Lie Algebra so(3)
Parameter Estimate

Exp. Map

$\mathbf{R} \in \mathrm{SO}(3)$ $\qquad$ $\boldsymbol{\omega}_{\times} \in \mathrm{so}(3)$

Log Map

${}_{\mathcal{A}}\boldsymbol{r}_{\mathcal{BC}}$ - A vector from B to C, expressed in frame A

$\Phi_{\mathcal{BA}} \in SO(3)$ - A rotation that maps points from frame A to frame B

${}_{\mathcal{B}}\boldsymbol{r}_{\mathcal{BC}} = \Phi_{\mathcal{BA}}\left({}_{\mathcal{A}}\boldsymbol{r}_{\mathcal{BC}}\right)$ - Co-ordinate mapping

${}_{\mathcal{B}}\boldsymbol{r} = \Phi_{\mathcal{BA}}({}_{\mathcal{A}}\boldsymbol{r})$ - Dropping Subscripts

$\boldsymbol{C} : SO(3) \rightarrow \mathbb{R}^{3\times 3}$ - Extraction of the Rotation matrix

Recall we have basis vectors e_i which in skew symmetric form are **Generators** for SO(3)

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad G_1 = \mathbf{e}_1^{\times} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad G_2 = \mathbf{e}_2^{\times} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

$$\mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad G_3 = \mathbf{e}_3^{\times} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\omega \in \mathbb{R}^3$$

$$\omega_1 G_1 + \omega_2 G_2 + \omega_3 G_3 \in so(3)$$

Skew symmetric matrices have certain properties:

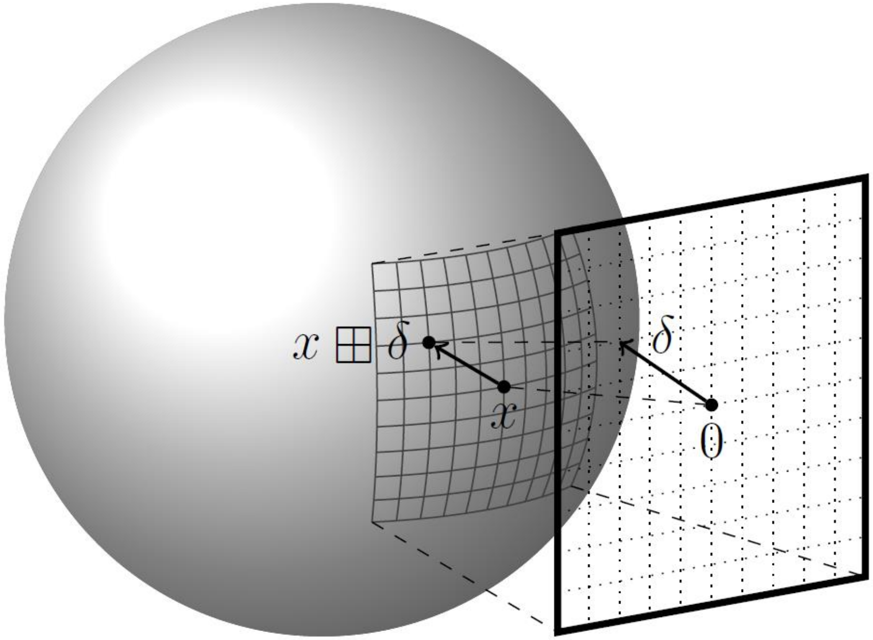
$$(\boldsymbol{v}^\times)^T = -\boldsymbol{v}^\times,$$
$$(\boldsymbol{v}^\times)^2 = \boldsymbol{v}\boldsymbol{v}^T - \boldsymbol{v}^T\boldsymbol{v}\boldsymbol{I},$$
$$(\boldsymbol{C}(\Phi)\boldsymbol{v})^\times = \boldsymbol{C}(\Phi)\boldsymbol{v}^\times\boldsymbol{C}(\Phi)^T$$

The vector cross product can be written using skew symmetric matrices.
Note that the cross product is **anticommutative** :

$$a \times b = a^\times b$$
$$a \times b = -b^\times a$$

$$\boxplus : \mathcal{S} \times \mathbb{R}^n \rightarrow \mathcal{S},$$

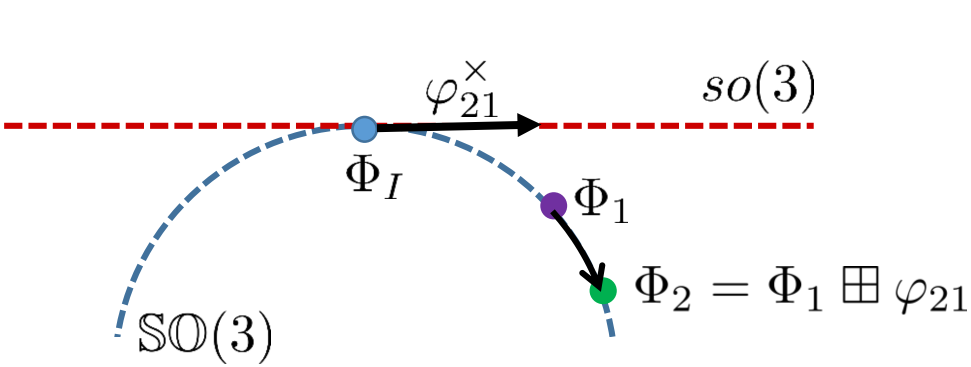
$$\boxminus : \mathcal{S} \times \mathcal{S} \rightarrow \mathbb{R}^n$$



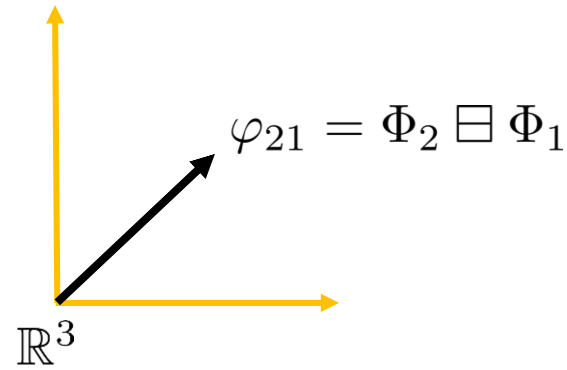
Specific implementation for SO(3), let's go through it on the board

$$\begin{aligned}\boxplus :& SO(3) \times \mathbb{R}^3 \to SO(3),\\ & \Phi, \boldsymbol{\varphi} \mapsto \exp(\boldsymbol{\varphi}) \circ \Phi,\\ \boxminus :& SO(3) \times SO(3) \to \mathbb{R}^3,\\ & \Phi_1, \Phi_2 \mapsto \log(\Phi_1 \circ \Phi_2^{-1})\end{aligned}$$

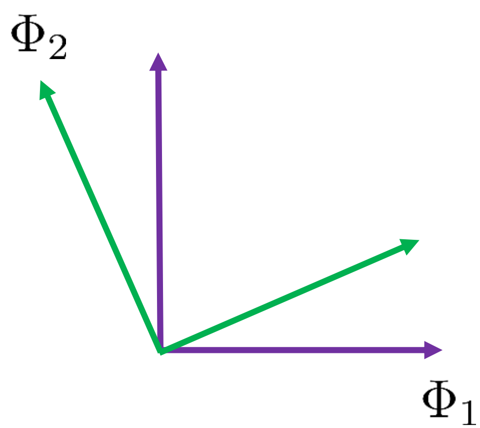
Specific implementation for SO(3), let's go through it on the board



Box-plus



Box-Minus

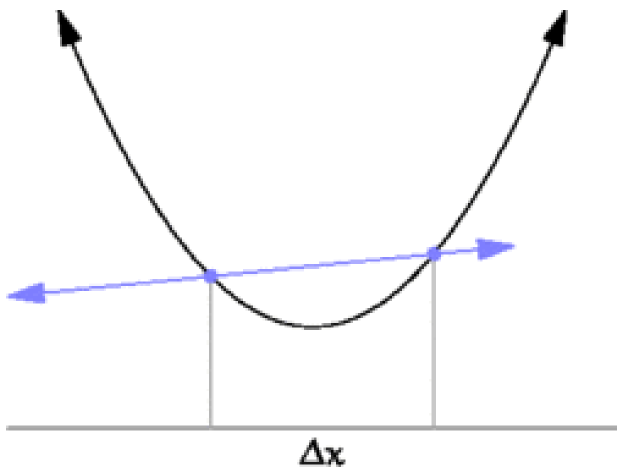


Physical World

Fairly simple to take the derivative of a function using the first principles definition:

$$f_1 : \mathbb{R} \mapsto \mathbb{R}$$

$$\frac{\partial f_1}{\partial x} = \lim_{\epsilon \to 0} \frac{f_1(x+\epsilon) - f_1(x)}{\epsilon}$$



When the function image is a manifold quantity, perform something similar to the easy case, except use box-minus to subtract group elements:

$$f_2 : \mathbb{R} \mapsto \mathbb{SO}(3)$$

$$\frac{\partial f_2}{\partial x} = \lim_{\epsilon \to 0} \frac{f_2(x+\epsilon) \boxminus f_2(x)}{\epsilon}$$

When the function domain is a manifold quantity, perform something similar to the previous case, except use box-plus for the function arguments:

$$f_3 : \mathbb{SO}(3) \mapsto \mathbb{R}$$

$$\frac{\partial f_3}{\partial \Phi} = \lim_{\epsilon \to 0} \begin{bmatrix} \frac{f_3(\Phi \boxplus (\mathbf{e}_1 \epsilon)) - f_3(\Phi)}{\epsilon} \\ \frac{f_3(\Phi \boxplus (\mathbf{e}_2 \epsilon)) - f_3(\Phi)}{\epsilon} \\ \frac{f_3(\Phi \boxplus (\mathbf{e}_3 \epsilon)) - f_3(\Phi)}{\epsilon} \end{bmatrix}$$

When the function domain and image is a manifold quantity, perform something similar to the previous case, except use box-minus and box-plus:

$$f_3 : \mathbb{SO}(3) \mapsto \mathbb{SO}(3)$$

$$\frac{\partial f_3}{\partial \Phi} = \lim_{\epsilon \to 0} \begin{bmatrix} \frac{f_3(\Phi \boxplus (\mathbf{e}_1 \epsilon)) \boxminus f_3(\Phi)}{\epsilon} \\ \frac{f_3(\Phi \boxplus (\mathbf{e}_2 \epsilon)) \boxminus f_3(\Phi)}{\epsilon} \\ \frac{f_3(\Phi \boxplus (\mathbf{e}_3 \epsilon)) \boxminus f_3(\Phi)}{\epsilon} \end{bmatrix}$$

Co-ordinate Mapping: $$\Phi_{\mathcal{B}\mathcal{A}}({}_{\mathcal{A}}r) = R_{\mathcal{B}\mathcal{A}}({}_{\mathcal{A}}r)$$

Composition: $$\Phi_1 \cdot \Phi_2$$

Inverse: $$\Phi_1^{-1}$$

We can build many residual functions just using these operations!

(1) Box-plus and box-minus

$$\boxplus : SO(3) \times \mathbb{R}^3 \rightarrow SO(3),$$
$$\Phi, \boldsymbol{\varphi} \mapsto \exp(\boldsymbol{\varphi}) \circ \Phi,$$
$$\boxminus : SO(3) \times SO(3) \rightarrow \mathbb{R}^3,$$
$$\Phi_1, \Phi_2 \mapsto \log(\Phi_1 \circ \Phi_2^{-1})$$

(2) Rodriguez Formula

$$C(\varphi) = C(\exp(\varphi))$$
$$= I + \frac{\sin(\|\varphi\|\varphi^{\times})}{\|\varphi\|} + \frac{(1-\cos(\|\varphi\|))\varphi^{\times 2}}{\|\varphi\|^2}$$
$$C(\varphi) \approx I + \varphi^{\times}, \quad (\|\varphi\| \approx 0)$$

(3) Inverse Identity

$$\exp(\varphi)^{-1} = \exp(-\varphi)$$

(4) Adjoint Related Identity

$$\exp(\Phi(\boldsymbol{\varphi})) = \Phi \circ \exp(\boldsymbol{\varphi}) \circ \Phi^{-1}$$

(5) anticommutative

$$a \times b = a^{\times} b$$
$$a \times b = -b^{\times} a$$

$$\left[\frac{\partial}{\partial \Phi}\Phi(\boldsymbol{r})\right]_i = \lim_{\epsilon \to 0} \frac{(\Phi \boxplus \boldsymbol{e}_i \epsilon)(\boldsymbol{r}) - \Phi(\boldsymbol{r})}{\epsilon}$$

$$
\begin{aligned}
\left[\frac{\partial}{\partial \Phi}\Phi(\boldsymbol{r})\right]_i &= \lim_{\epsilon \to 0} \frac{(\Phi \boxplus \boldsymbol{e}_i \epsilon)(\boldsymbol{r}) - \Phi(\boldsymbol{r})}{\epsilon} \\
&= \lim_{\epsilon \to 0} \frac{\boldsymbol{C}(\boldsymbol{e}_i \epsilon)\boldsymbol{C}(\Phi)\boldsymbol{r} - \boldsymbol{C}(\Phi)\boldsymbol{r}}{\epsilon} \\
&= \lim_{\epsilon \to 0} \frac{(\boldsymbol{I} + \boldsymbol{e}_i^{\times} \epsilon)\boldsymbol{C}(\Phi)\boldsymbol{r} - \boldsymbol{C}(\Phi)\boldsymbol{r}}{\epsilon} \\
&= \lim_{\epsilon \to 0} \frac{\boldsymbol{e}_i^{\times} \epsilon \boldsymbol{C}(\Phi)\boldsymbol{r}}{\epsilon} \\
&= -\left(\boldsymbol{C}(\Phi)\boldsymbol{r}\right)^{\times} \boldsymbol{e}_i .
\end{aligned}
$$

$$
\frac{\partial}{\partial \Phi}\Phi(\boldsymbol{r}) = -\left(\boldsymbol{C}(\Phi)\boldsymbol{r}\right)^{\times} .
$$

$$\left[\frac{\partial}{\partial \Phi_1} \Phi_1 \circ \Phi_2\right]_i = \lim_{\epsilon \to 0} \frac{((\Phi_1 \boxplus \boldsymbol{e}_i \epsilon) \circ \Phi_2) \boxminus (\Phi_1 \circ \Phi_2)}{\epsilon}$$

$$\left[\frac{\partial}{\partial \Phi_1} \Phi_1 \circ \Phi_2\right]_i = \lim_{\epsilon \to 0} \frac{((\Phi_1 \boxplus \boldsymbol{e}_i \epsilon) \circ \Phi_2) \boxminus (\Phi_1 \circ \Phi_2)}{\epsilon}$$

$$= \lim_{\epsilon \to 0} \frac{\log(\exp(\boldsymbol{e}_i \epsilon) \circ \Phi_1 \circ \Phi_2 \circ \Phi_2^{-1} \circ \Phi_1^{-1})}{\epsilon}$$

$$= \boldsymbol{e}_i.$$

$$\frac{\partial}{\partial \Phi_1} \Phi_1 \circ \Phi_2 = \boldsymbol{I}.$$

$$\left[\frac{\partial}{\partial \Phi_2} \Phi_1 \circ \Phi_2\right]_i = \lim_{\epsilon \to 0} \frac{(\Phi_1 \circ (\Phi_2 \boxplus \boldsymbol{e}_i \epsilon)) \boxminus (\Phi_1 \circ \Phi_2)}{\epsilon}$$

$$\left[\frac{\partial}{\partial \Phi_2}\Phi_1 \circ \Phi_2\right]_i = \lim_{\epsilon \to 0} \frac{(\Phi_1 \circ (\Phi_2 \boxplus \boldsymbol{e}_i \epsilon)) \boxminus (\Phi_1 \circ \Phi_2)}{\epsilon}$$

$$= \lim_{\epsilon \to 0} \frac{\log(\Phi_1 \circ \exp(\boldsymbol{e}_i \epsilon) \circ \Phi_2 \circ \Phi_2^{-1} \circ \Phi_1^{-1})}{\epsilon}$$

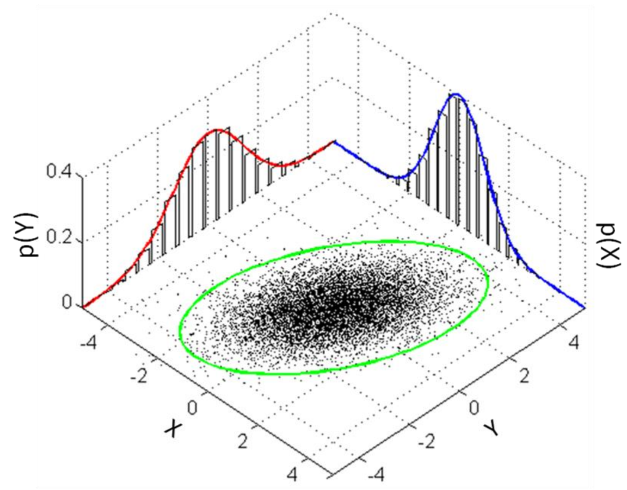
$$= \lim_{\epsilon \to 0} \frac{\log(\exp(\Phi_1(\boldsymbol{e}_i)\epsilon))}{\epsilon}$$

$$= \Phi_1(\boldsymbol{e}_i) = \boldsymbol{C}(\Phi_1)\boldsymbol{e}_i.$$

$$\frac{\partial}{\partial \Phi_2}\Phi_1 \circ \Phi_2 = \boldsymbol{C}(\Phi_1).$$

Gaussian random variables take the form:

$$\mathbf{x} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$$



Often, we write them as some nominal value with zero mean noise:

$$\mathbf{x} = \boldsymbol{\mu} + \boldsymbol{\epsilon}, \quad \boldsymbol{\epsilon} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma})$$

This works because **x** lives in a vector space (closed under +). What should we do for SO(3)?

There are a few options for SO(3), either performing perturbations on the Lie Group, or the Lie Algebra:

	$SO(3)$	$\mathfrak{so}(3)$
left	$\mathbf{C} = \exp\left(\boldsymbol{\epsilon}_\ell^\wedge\right)\bar{\mathbf{C}}$	~~$\boldsymbol{\phi} \approx \boldsymbol{\mu} + \mathbf{J}_\ell(\boldsymbol{\mu})\,\boldsymbol{\epsilon}_\ell$~~
middle	$\mathbf{C} = \exp\left((\boldsymbol{\mu} + \boldsymbol{\epsilon}_m)^\wedge\right)$	$\boldsymbol{\phi} = \boldsymbol{\mu} + \boldsymbol{\epsilon}_m$
right	$\mathbf{C} = \bar{\mathbf{C}}\exp\left(\boldsymbol{\epsilon}_r^\wedge\right)$	~~$\boldsymbol{\phi} \approx \boldsymbol{\mu} + \mathbf{J}_r(\boldsymbol{\mu})\,\boldsymbol{\epsilon}_r$~~

Pros and cons?

There are a few options for SO(3), either performing perturbations on the Lie Group, or the Lie Algebra:

	$SO(3)$	$\mathfrak{so}(3)$
left	$\mathbf{C} = \exp\left(\boldsymbol{\epsilon}_\ell^\wedge\right)\bar{\mathbf{C}}$	~~$\boldsymbol{\phi} \approx \boldsymbol{\mu} + \mathbf{J}_\ell(\boldsymbol{\mu})\,\boldsymbol{\epsilon}_\ell$~~
middle	$\mathbf{C} = \exp\left((\boldsymbol{\mu} + \boldsymbol{\epsilon}_m)^\wedge\right)$	$\boldsymbol{\phi} = \boldsymbol{\mu} + \boldsymbol{\epsilon}_m$
right	$\mathbf{C} = \bar{\mathbf{C}}\exp\left(\boldsymbol{\epsilon}_r^\wedge\right)$	~~$\boldsymbol{\phi} \approx \boldsymbol{\mu} + \mathbf{J}_r(\boldsymbol{\mu})\,\boldsymbol{\epsilon}_r$~~

Generally prefer perturbations in Lie Group to avoid singularities

$$\mathbf{C} = \exp\left(\boldsymbol{\epsilon}^\wedge\right)\bar{\mathbf{C}}$$

Consider:

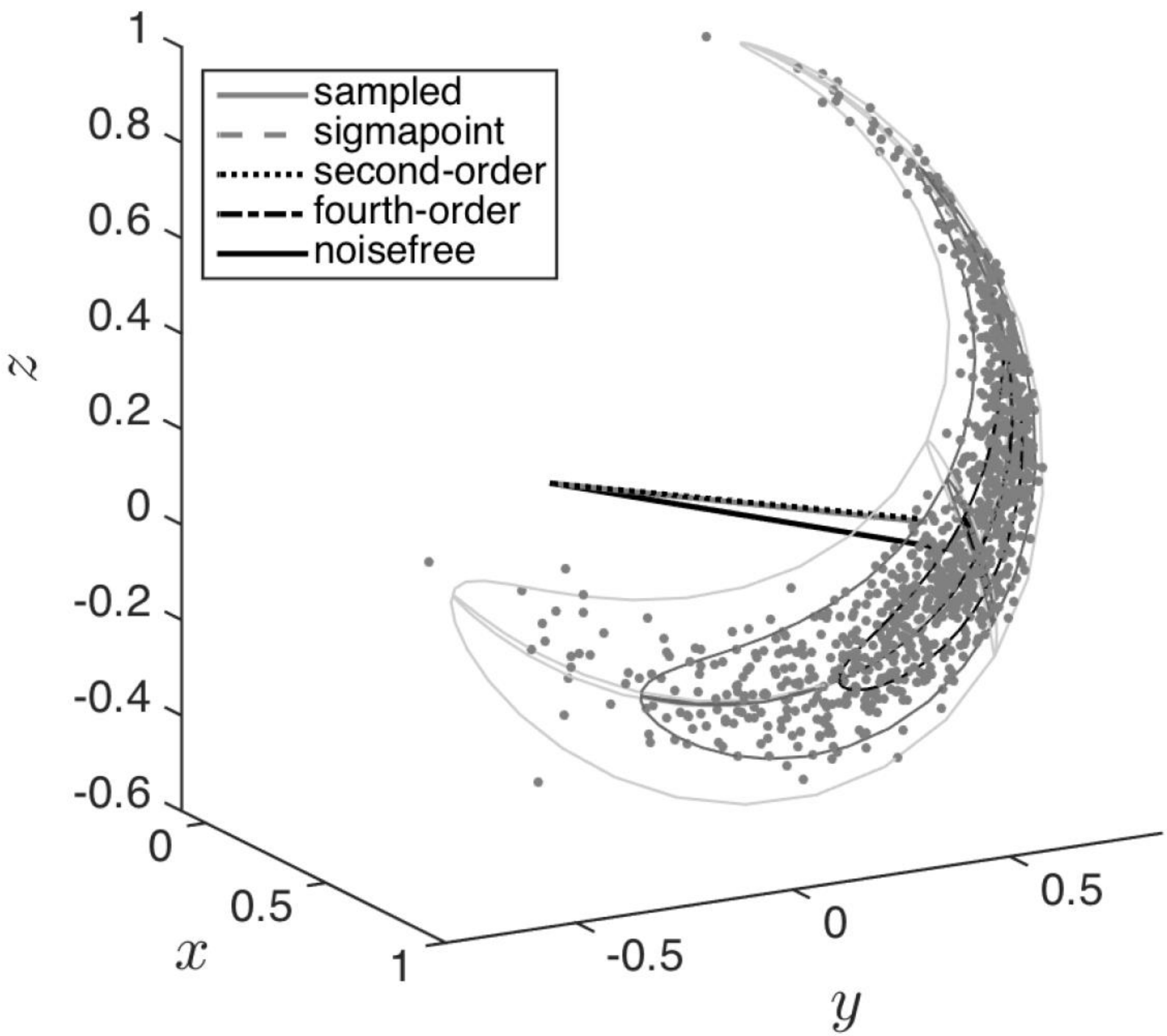
$$\mathbf{y} = \mathbf{C}\mathbf{x}$$

$$\mathbf{C} = \exp\left(\boldsymbol{\epsilon}^{\wedge}\right)\bar{\mathbf{C}}, \quad \boldsymbol{\epsilon} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma})$$

What does the distribution over **y** look like?

$$\mathbf{y} = \mathbf{C}\mathbf{x}$$

$$\mathbf{C} = \exp\left(\boldsymbol{\epsilon}^{\wedge}\right)\bar{\mathbf{C}},$$
$$\boldsymbol{\epsilon} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma})$$



What about E[**y**]? We can perform an expansion of the exponential map term:

$$\mathbf{y} = \mathbf{C}\mathbf{x} = \left(\mathbf{1} + \epsilon^\wedge + \frac{1}{2}\epsilon^\wedge\epsilon^\wedge + \frac{1}{6}\epsilon^\wedge\epsilon^\wedge\epsilon^\wedge + \frac{1}{24}\epsilon^\wedge\epsilon^\wedge\epsilon^\wedge\epsilon^\wedge + \cdots\right)\bar{\mathbf{C}}\mathbf{x}$$

$$E[\mathbf{y}] = \left(\mathbf{1} + \frac{1}{2}E\left[\epsilon^\wedge\epsilon^\wedge\right] + \frac{1}{24}E\left[\epsilon^\wedge\epsilon^\wedge\epsilon^\wedge\epsilon^\wedge\right] + \cdots\right)\bar{\mathbf{C}}\mathbf{x}.$$

$$\vdots$$

$$E[\mathbf{y}] \approx \Bigg(\mathbf{1} + \frac{1}{2}\left(-\operatorname{tr}\left(\boldsymbol{\Sigma}\right)\mathbf{1} + \boldsymbol{\Sigma}\right) + \frac{1}{24}\left(\left(\left(\operatorname{tr}\left(\boldsymbol{\Sigma}\right)\right)^2 + 2\operatorname{tr}\left(\boldsymbol{\Sigma}^2\right)\right)\mathbf{1} - \boldsymbol{\Sigma}\left(\operatorname{tr}\left(\boldsymbol{\Sigma}\right)\mathbf{1} + 2\boldsymbol{\Sigma}\right)\right)\Bigg)\bar{\mathbf{C}}\mathbf{x}$$

(Fourth order expansion)

Homework: Derive the following expression for the derivative of the inverse mapping

$$\left[\frac{\partial}{\partial \Phi} \Phi^{-1}\right]_i =$$