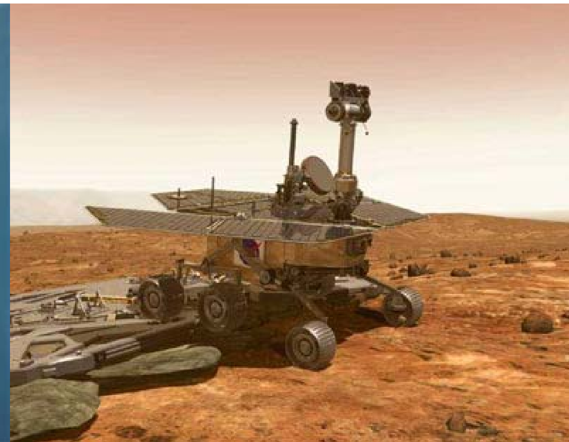
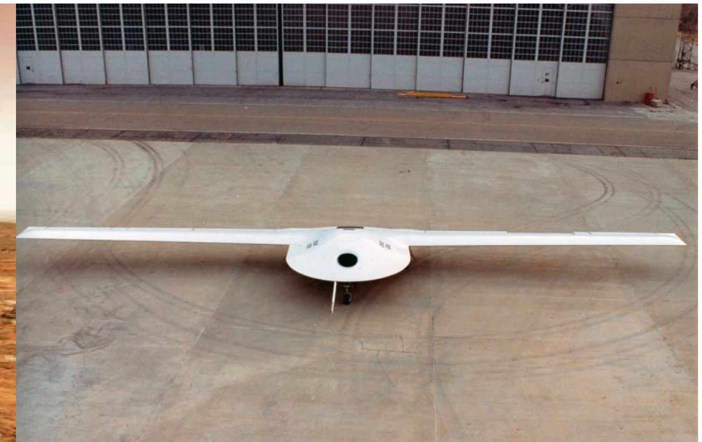
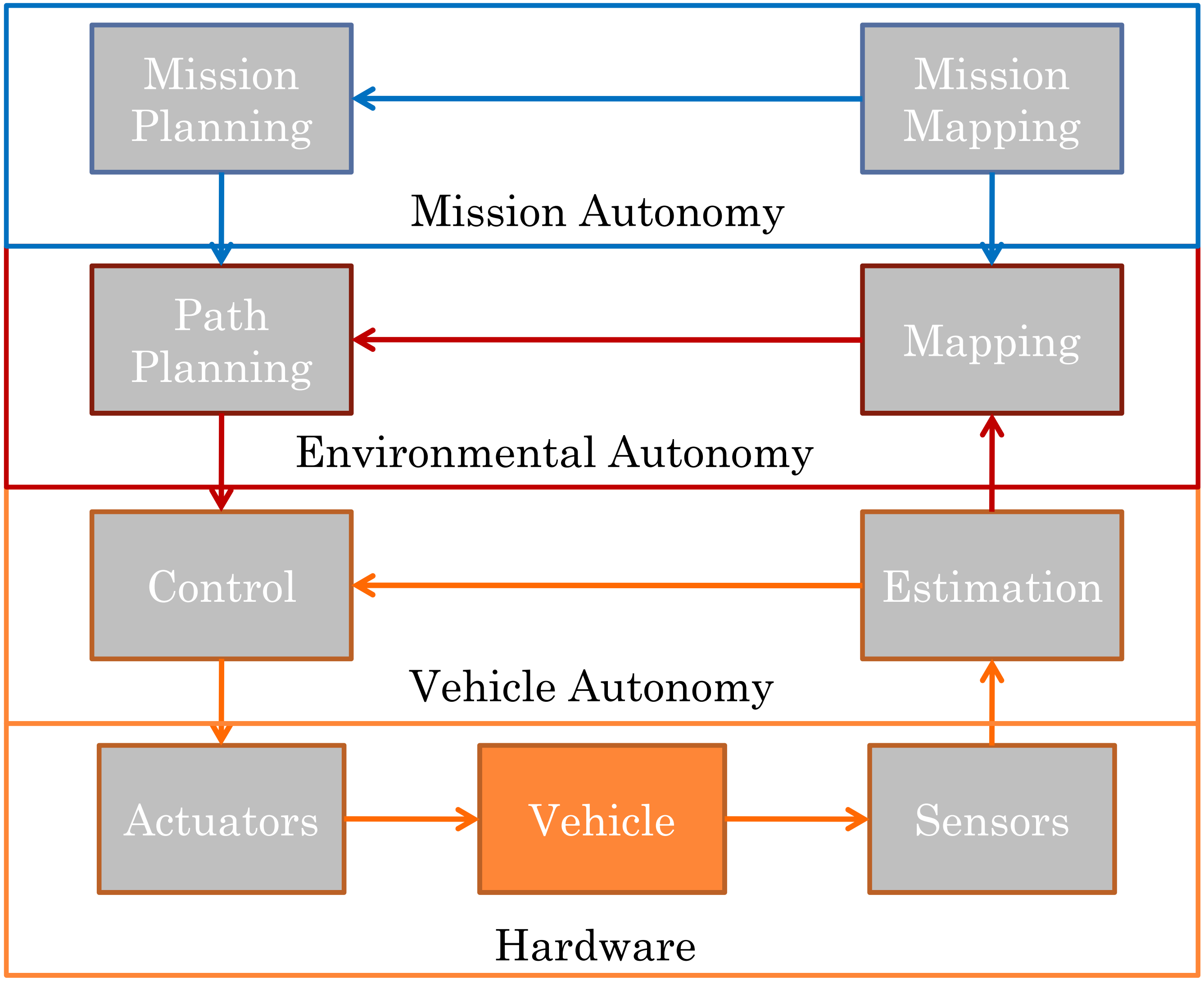




ME 597: Autonomous Mobile Robotics
Section 3 – Motion Modeling

Prof. Steven Waslander

Components



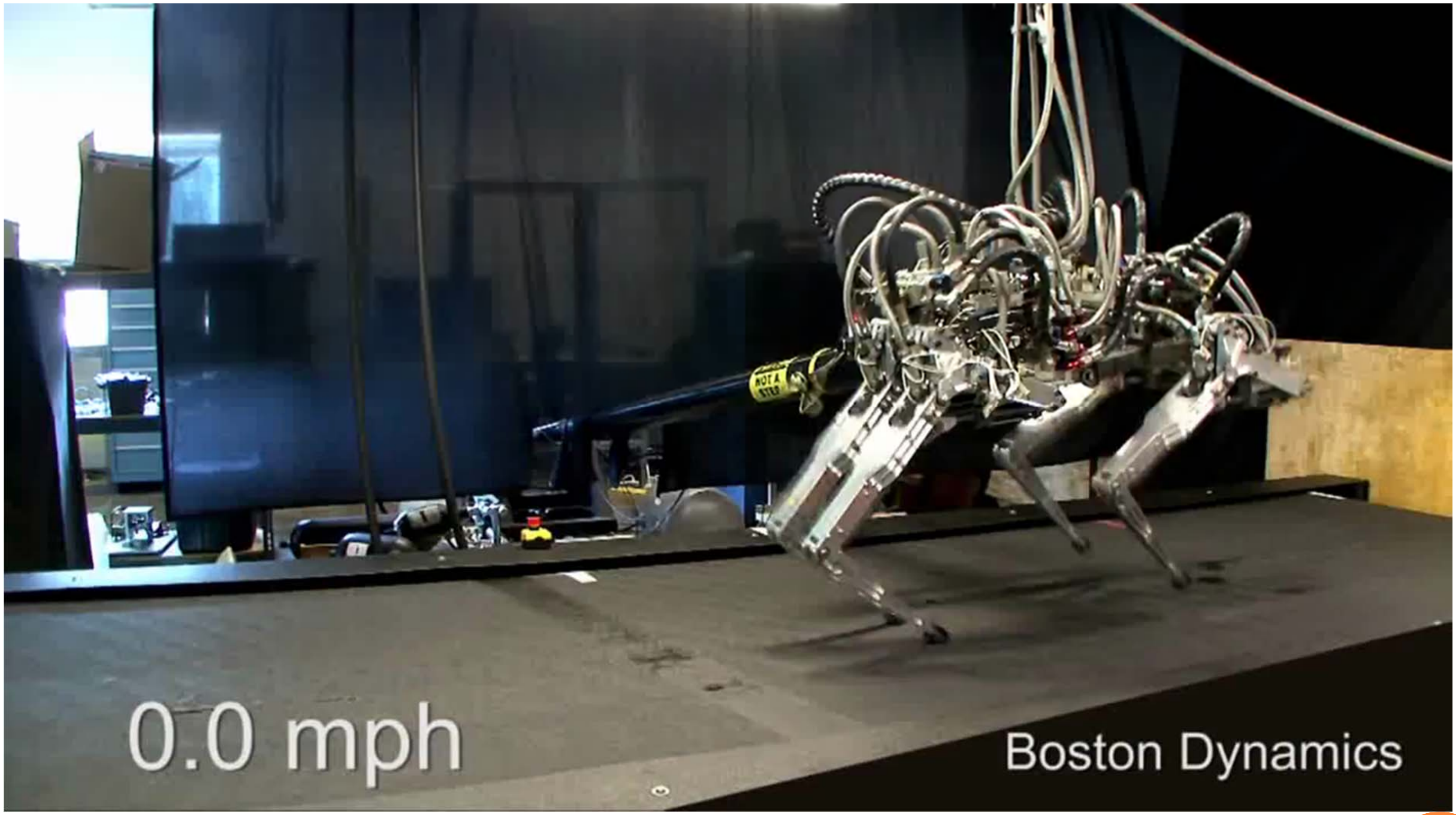
Mission Planning
Mission Mapping
Mission Autonomy
Path Planning
Mapping
Environmental Autonomy
Control
Estimation
Vehicle Autonomy
Actuators
Vehicle
Sensors
Hardware

Outline

- Example Videos
  - Wheeled, Legged, Aerial, Aquatic
- Motion Modeling
  - Definitions
  - Kinematics and Dynamics
  - Standard models and disturbances

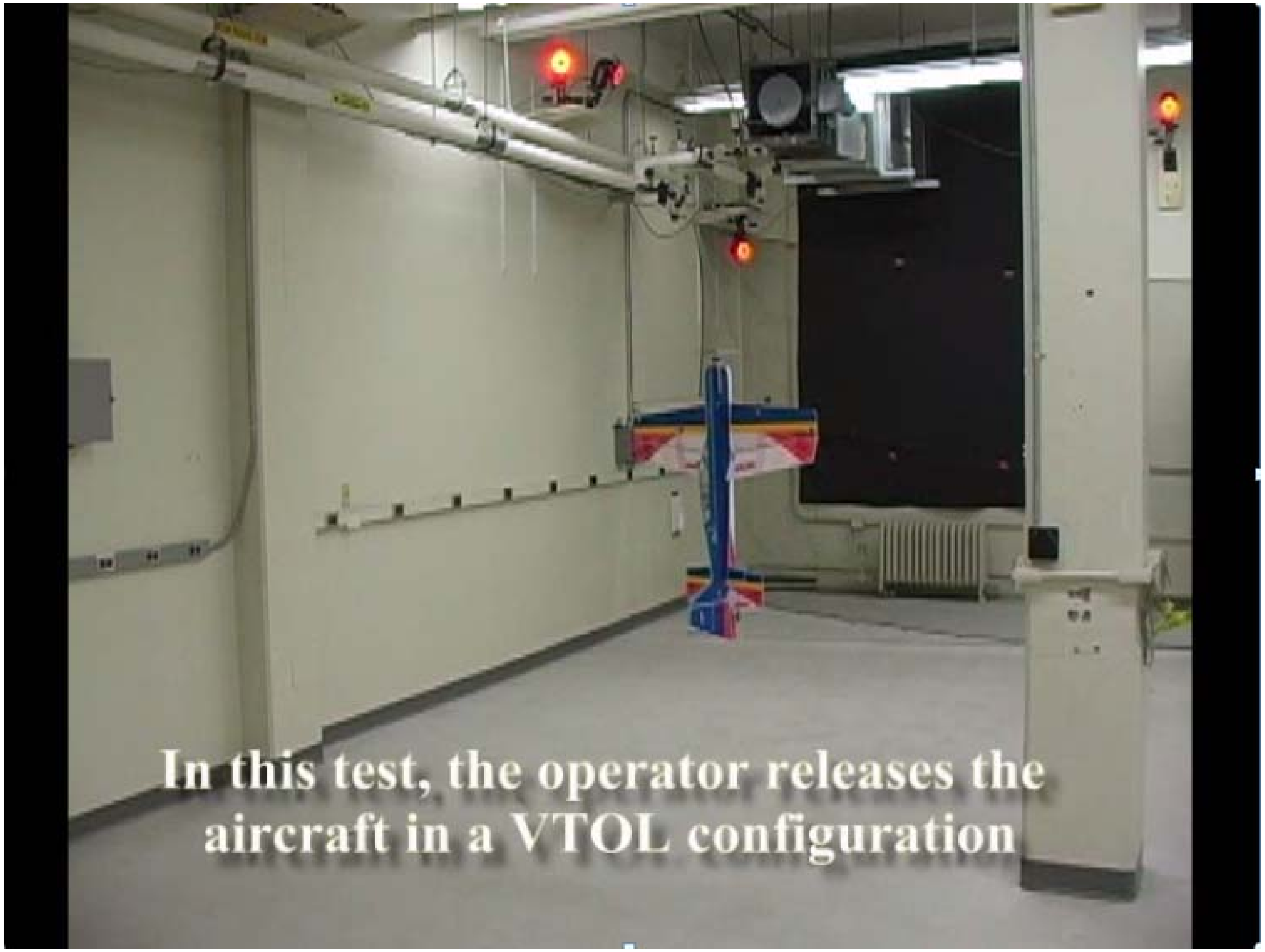
Swedish Wheels in Action

Legged Robots in Action



Legged Robots in Action

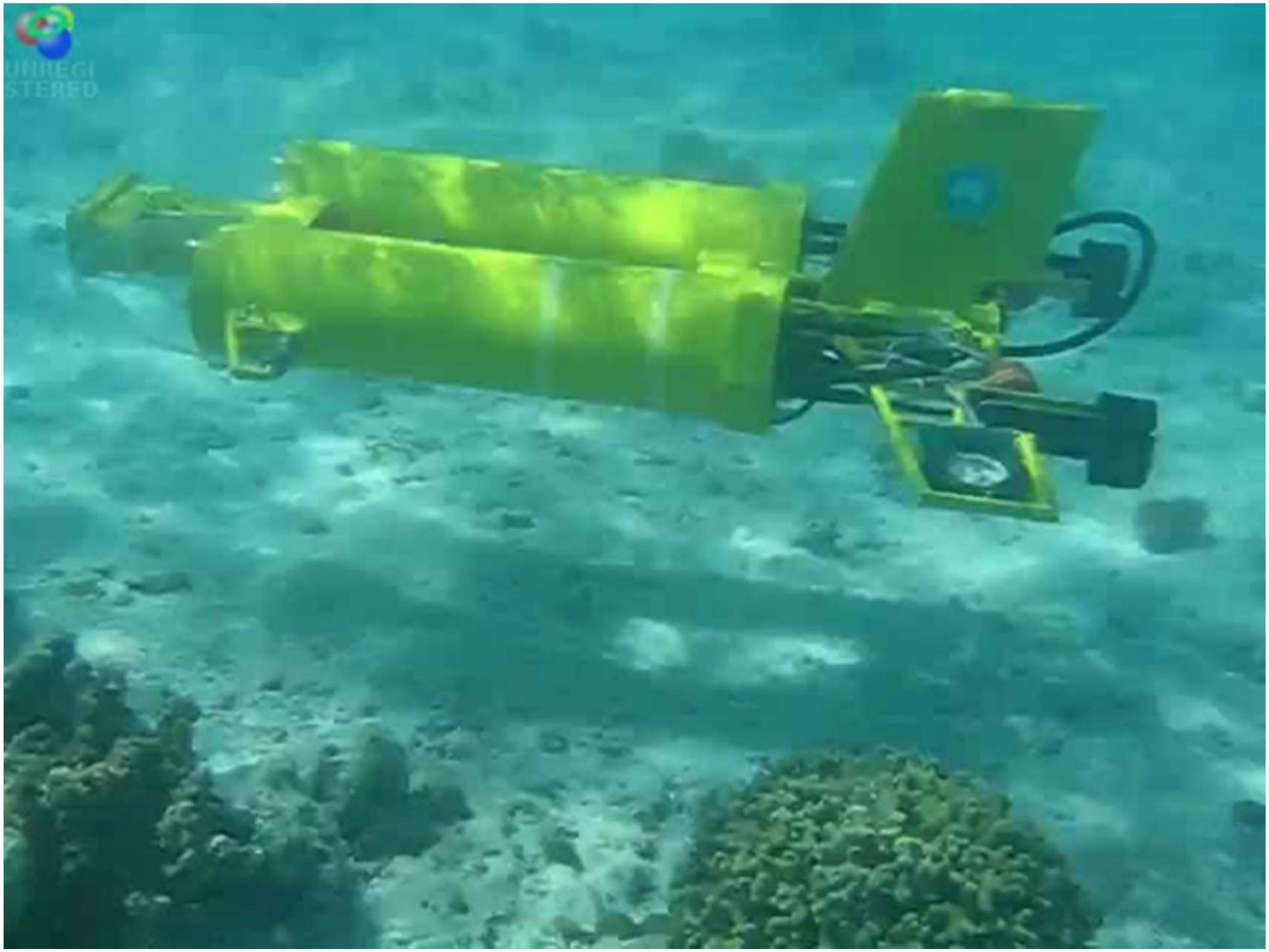
Aerial Robots In Action



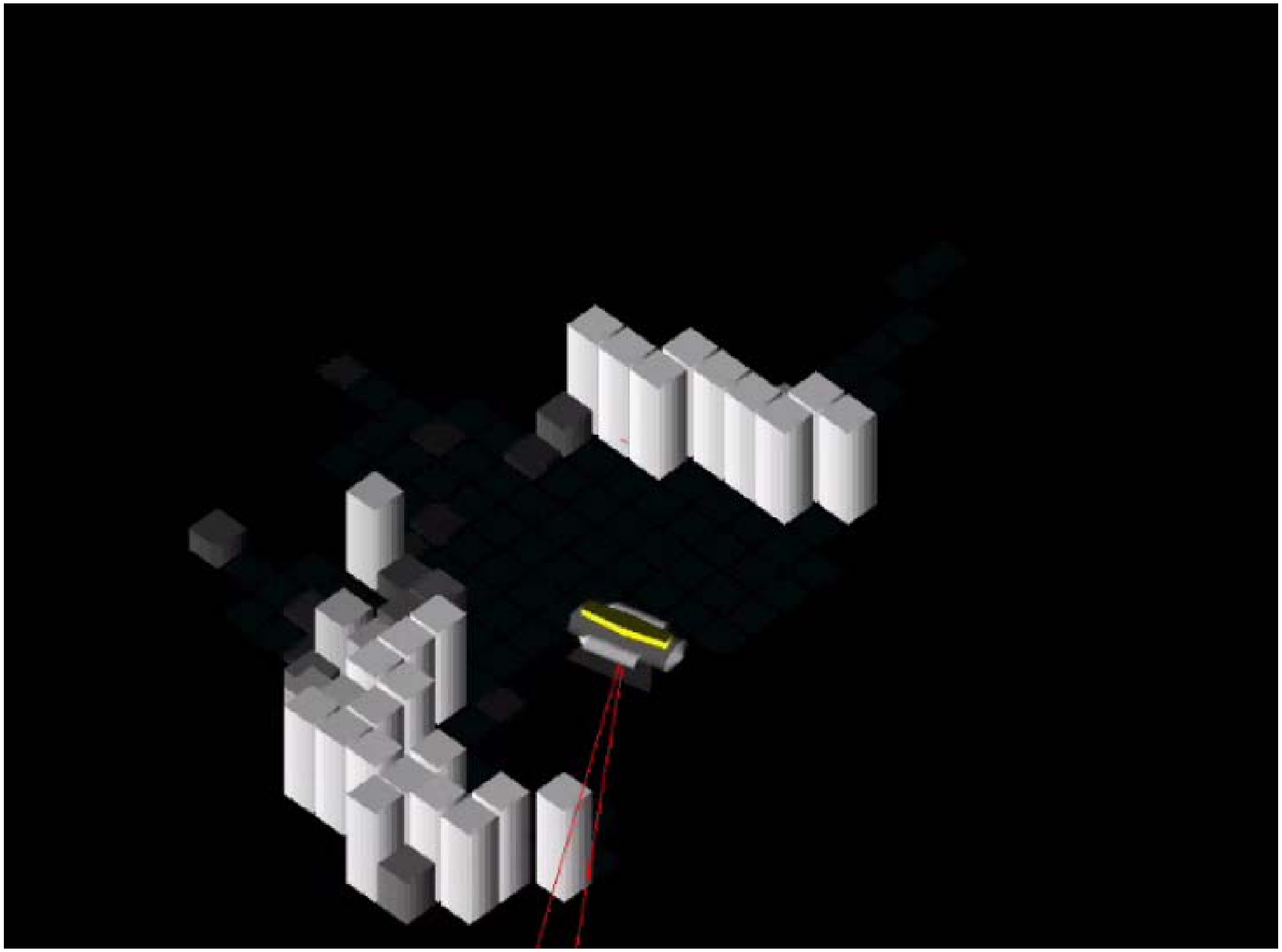
Aerial Robots In Action

Aerial Robots In Action

Aquatic Robots in Action



Aquatic Robots in Action



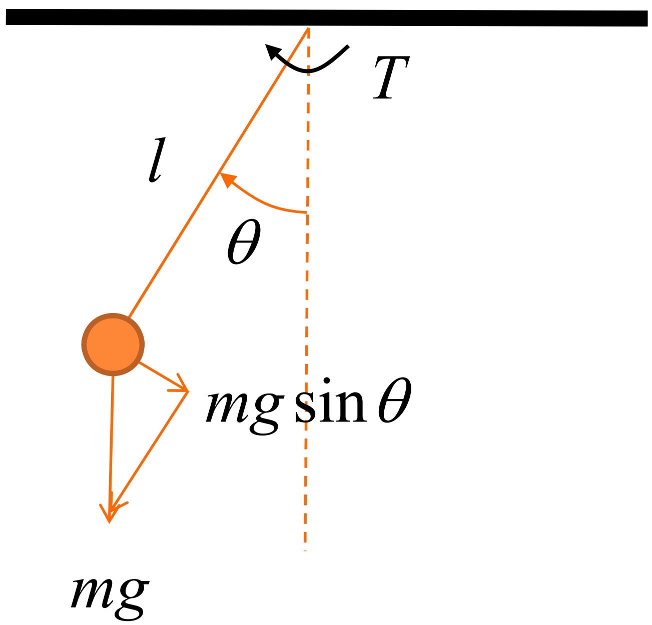
Aquatic Robots in Action

Outline

- Example Videos
  - Wheeled, Legged, Aerial, Aquatic
- Motion Modeling
  - Definitions
  - Kinematics and Dynamics
  - Standard models and disturbances

Motion Modeling

- A motion model seeks to describe how system motion can occur
  - Given inputs (*T*), what will the system do (*Θ*)?
  - Define a set of constraints between states and inputs
  - Define unknown disturbances as distributions (ε)



$$ml^2\ddot{\theta} + mgl\sin\theta = T + \varepsilon$$

State

Recall: The ***state*** of a system is a vector of system variables that entirely defines the system at a specific instance in time.

- The state may include variables describing only the vehicle, or the vehicle and it's environment

  - Common Vehicle States: Position, Velocity, Attitude, Attitude Rates, Motor Speeds, Battery/Fuel Level

  - Common Environment States: Feature Locations, Surface Polygons and Normals, Wind Conditions, Ocean Currents

Complete State

Definition: A state vector is **complete** if it is the best predictor of the future.

- **Known as the Markov Assumption**
  - Ensures that past and future states are independent if the current state is known
- Very useful for estimation and control
  - No need to store excessive amounts of data, only the current state
- Must balance size of model (number of states) with violations of Markov assumption
  - Ignoring the state of wind on an aircraft leads to significant errors in velocity control
  - However, integral control can accommodate

INPUTS

Definition: The ***inputs*** of a system are the set of variables that drive the system that can be controlled.

- The same notion as in classical and state space controls
- Common vehicle inputs:
  - Motor throttle, voltage, servo pwm command, steering angle, elevator angle
- Referred to as control actions in Thrun et al.

Disturbances

Definition: The ***disturbances*** of a system are the set of variables that drive the system that cannot be controlled.

- Disturbances are why we need a control system (and stability), cause uncertainty in state
- The better the model of disturbances, the better their effect can be rejected.
- Most common model used is additive Gaussian
  - Often then augmented with linear, nonlinear mapping

Motion Model

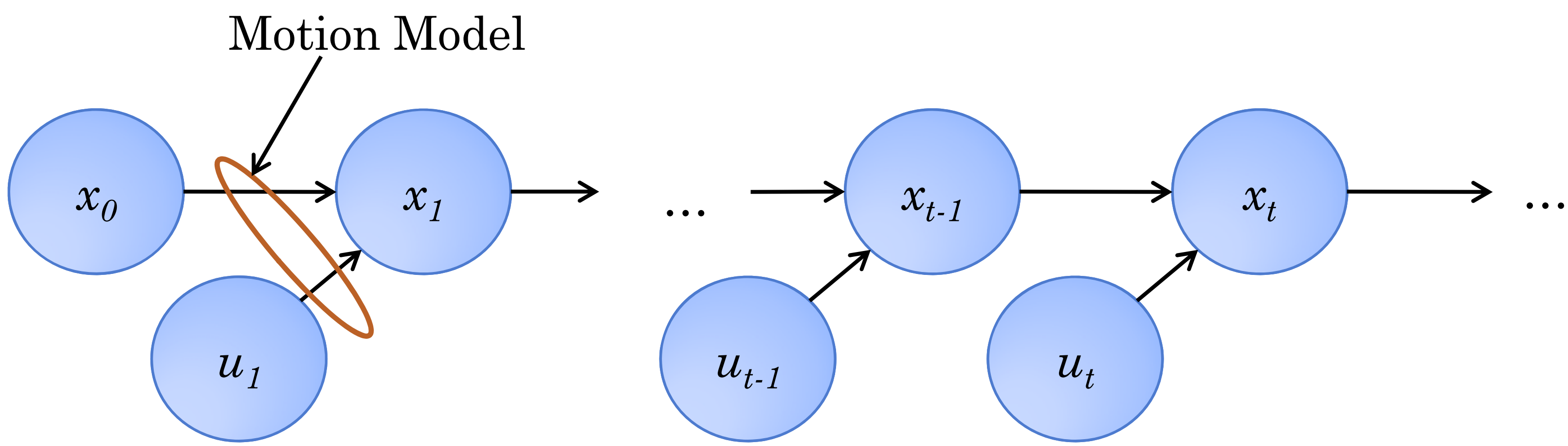
- The discrete time motion model is defined as

$$x_t = f(x_{t-1}, u_t, \varepsilon_t)$$

  - x_t is the state vector at time t
  - u_t is the input vector at time t
  - $\mathcal{E}_t$ is the disturbance to the system at time t
    - *e.g.* $$\varepsilon_t \sim N(0, R_t)$$
  - $f(x_{t-1}, u_t, \mathcal{E}_t)$ is the motion model
    - Can be linear, nonlinear, discontinuous, hybrid

Motion Model

- The Markov assumption implies the following dependencies
- The motion model captures a single transition from one time period to the next



Motion Model Types

- Linear models with additive Gaussian disturbances
  - Linear models satisfy superposition
  - Can always be written in standard form

$$x_t = Ax_{t-1} + Bu_t + \varepsilon_t$$

- Nonlinear models with additive Gaussian disturbances

$$x_t = f(x_{t-1}, u_t) + \varepsilon_t$$

- Nonlinear models with nonlinear disturbances

$$x_t = f(x_{t-1}, u_t, \varepsilon_t)$$

Motion Model Types

- Motion models can be viewed in other ways
  - Probabilistically, the probability of ending in state x_t given input u_t and prior state x_{t-1}

$$p(x_t \mid x_{t-1}, u_t)$$

  - Continuous time, the rate of change of the states is governed by a nonlinear function of state, input and disturbances

$$\dot{x} = \bar{f}(x, u, \varepsilon)$$

Motion Models

- Degrees of freedom vs. states
  - Degrees of freedom are axes of independent motion, whereas states may also include derivatives and other terms.
    - A quadrotor has 6 degrees of freedom (X, Y, Z, roll, pitch, yaw) but at least 12 states.

- Constraints on the motion
  - Holonomic vs. nonholonomic constraints
    - Holonomic constraints depend only on the "position" of the vehicle
      - On the states that define the degrees of freedom
    - Nonholonomic constraints also depend on velocity (or the derivatives of the position)
      - On all states, on how a vehicle moves

Kinematic and Dynamic Models

- At low speeds, it is often sufficient to look only at kinematic models of vehicles
  - Two wheeled robot
  - Bicycle model
- However, when forces vary with the state, more precise modeling can be beneficial
  - Dynamic modeling of cars for cruise control
  - Quadrotor dynamics
- In this course, models will mostly be supplied, assume you already know how to define them
  - These notes now cover the basic models we use

Motion Models

- Five basic models to be used throughout course
  - Linear dynamic model
    - AUV
  - Simple 2D Nonholonomic model
    - Two-wheel robot, speed and rotation rate inputs
  - 2D Nonholonomic model
    - Two-wheel robot, left and right wheel speed inputs
  - Bicycle model
    - Two wheel model with speed and steering angle
    - Valid for four wheel cars as well
  - Quadrotor dynamic model
    - Example of 6DOF model with four thrust inputs

Linear Dynamic Model

- Linear Example – Simple AUV
  - 3D Linear motion model for three thruster AUV (attitude held constant)
    - State

$$x = \begin{bmatrix} p_n \\ v_n \\ p_e \\ v_e \\ p_d \\ v_d \end{bmatrix}$$

    - Input

$$u = \begin{bmatrix} T_n \\ T_e \\ T_d \end{bmatrix}$$

    - Continuous dynamics for

$$\dot{x}_n(t) = v_n(t)$$
$$m\dot{v}_n(t) = -bv_n(t) + T_n(t)$$

Linear Dynamic Model

- Example – Linear AUV
  - Should always perform discretization through zero-order hold, first-order hold, Tustins
  - Simple alternative (for exams, proof of concept):
    - Approximating left hand side derivatives with finite differences, holding right hand side at previous values

$$\frac{x_{n,t} - x_{n,t-1}}{dt} = v_{n,t-1}$$

$$m\frac{v_{n,t} - v_{n,t-1}}{dt} = -bv_{n,t-1} + T_{n,t}$$

  - Solving for current state

$$x_{n,t} = x_{n,t-1} + v_{n,t-1}dt$$

$$v_{n,t} = v_{n,t-1} - \frac{b}{m}v_{n,t-1}dt + \frac{T_{n,t}}{m}dt$$

Linear Dynamic Model

- Example – Linear AUV
  - Discrete Dynamics

$$x_t = \begin{bmatrix} 1 & dt & 0 & 0 & 0 & 0 \\ 0 & 1-b_n dt/m & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & dt & 0 & 0 \\ 0 & 0 & 0 & 1-b_e dt/m & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & dt \\ 0 & 0 & 0 & 0 & 0 & 1-b_d dt/m \end{bmatrix} x_{t-1} + \begin{bmatrix} 0 & 0 & 0 \\ dt/m & 0 & 0 \\ 0 & 0 & 0 \\ 0 & dt/m & 0 \\ 0 & 0 & 0 \\ 0 & 0 & dt/m \end{bmatrix} u_t$$

- Comes from approximation to matrix exponential

$$e^{At} \approx I + Adt$$

- More accurate approaches (which rely on actually calculating matrix exponential) can be used by taking advantage of built-in Matlab tools.

LINEAR DYNAMIC MODEL

- Example – Linear AUV
  - Discrete Dynamics from zero order hold of continuous model (only N,E directions for plotting purposes)

$$x_t = \begin{bmatrix} 1 & .0975 & 0 & 0 \\ 0 & .9512 & 0 & 0 \\ 0 & 0 & 1 & .0975 \\ 0 & 0 & 0 & .9512 \end{bmatrix} x_{t-1} + \begin{bmatrix} .0025 & 0 \\ .0488 & 0 \\ 0 & .0025 \\ 0 & .048 \end{bmatrix} u_t + \varepsilon_t$$

  - Comparison to above solution

$$x_t = \begin{bmatrix} 1 & .1 & 0 & 0 \\ 0 & .95 & 0 & 0 \\ 0 & 0 & 1 & .1 \\ 0 & 0 & 0 & .95 \end{bmatrix} x_{t-1} + \begin{bmatrix} .0 & 0 \\ .05 & 0 \\ 0 & .0 \\ 0 & .05 \end{bmatrix} u_t + \varepsilon_t$$

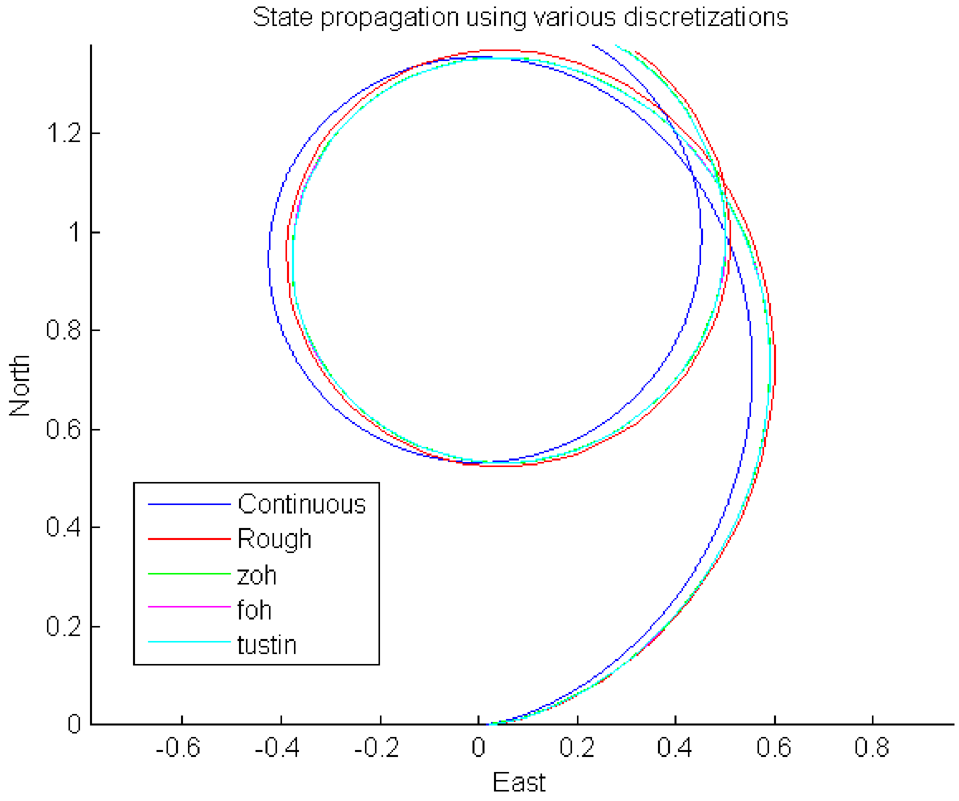
Linear Dynamic Model

- Example – Linear AUV in X,Y plane

$$t = 10$$

$$dt = 0.1$$

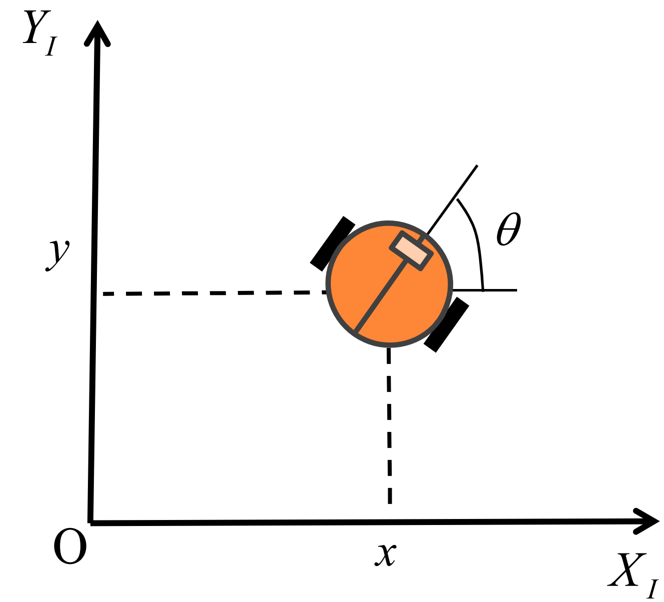
$$u = \begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix}$$



TWO-WHEELED KINEMATIC MODEL

- Two-wheeled robot
  - Vehicle State, Inputs:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x \\ y \\ \theta \end{bmatrix} \qquad \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} v \\ \omega \end{bmatrix}$$



  - Motion Model

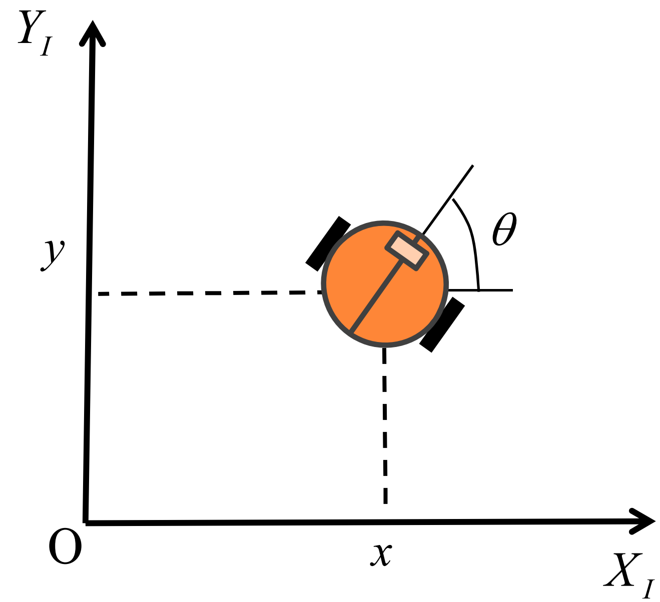
$$\begin{bmatrix} x_{1,t} \\ x_{2,t} \\ x_{3,t} \end{bmatrix} = g(x_{t-1}, u_t) = \begin{bmatrix} x_{1,t-1} + u_{1,t} \cos x_{3,t-1} dt \\ x_{2,t-1} + u_{1,t} \sin x_{3,t-1} dt \\ x_{3,t-1} + u_{2,t} dt \end{bmatrix}$$

Two-Wheeled Kinematic Model

- Two-wheeled robot with additive Gaussian disturbances
  - Disturbance model

    $$\varepsilon_t \sim N(0, R_t)$$

    - R_t diagonal
    - Independent disturbances

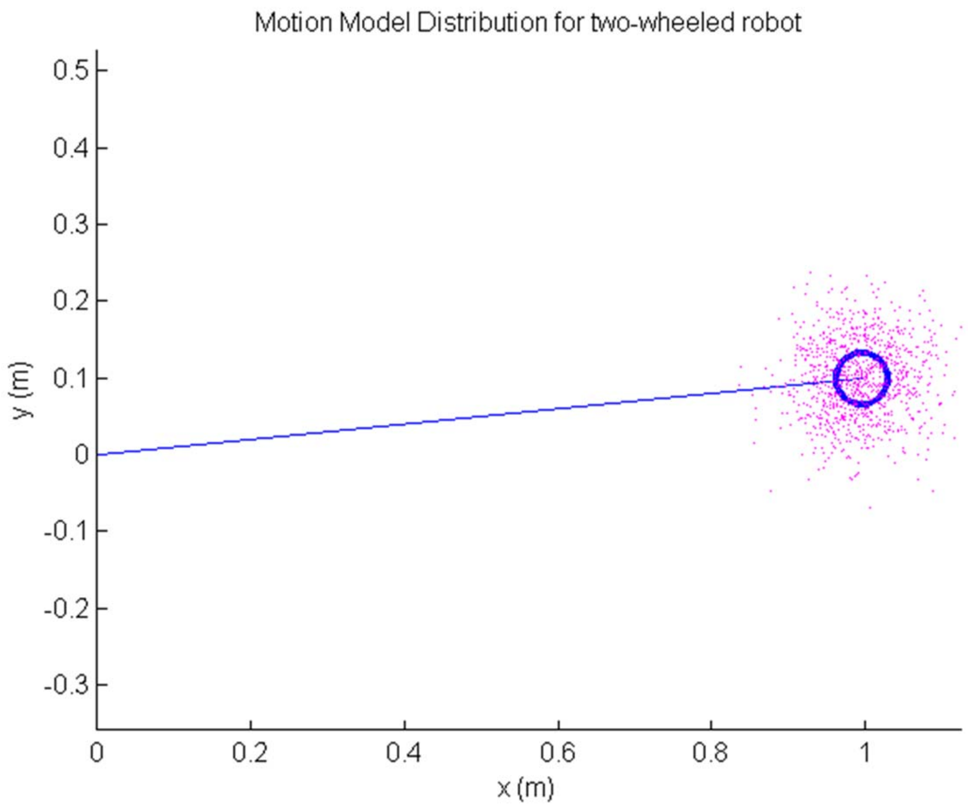


  - Motion Model

$$\begin{bmatrix} x_{1,t} \\ x_{2,t} \\ x_{3,t} \end{bmatrix} = g(x_{t-1}, u_t, \varepsilon_t) = \begin{bmatrix} x_{1,t-1} + u_{1,t} \cos x_{3,t-1} dt \\ x_{2,t-1} + u_{1,t} \sin x_{3,t-1} dt \\ x_{3,t-1} + u_{2,t} dt \end{bmatrix} + \varepsilon_t$$

Two-Wheeled Kinematic Model

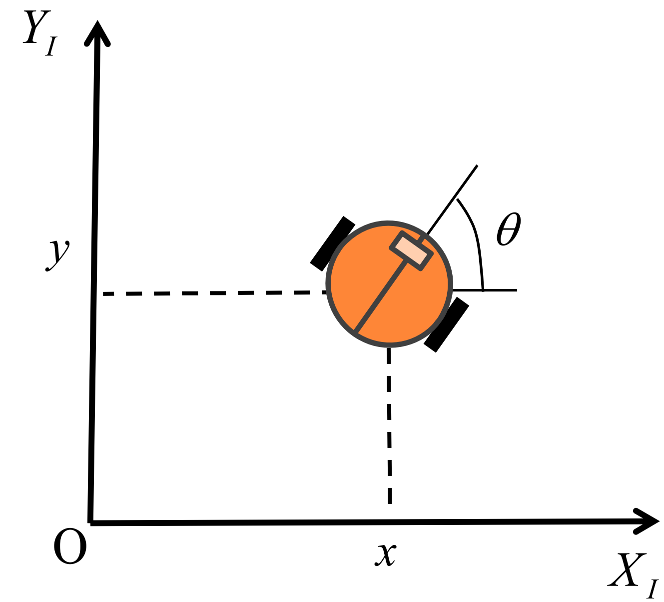
- Example: Two-wheeled robot with additive Gaussian disturbances



$$\begin{bmatrix} 0.05 & 0 & 0 \\ 0 & 0.05 & 0 \\ 0 & 0 & 0.1 \end{bmatrix}$$

Kinematic Model

- Two-wheeled robot with nonlinear disturbances
  - Speed, heading affected



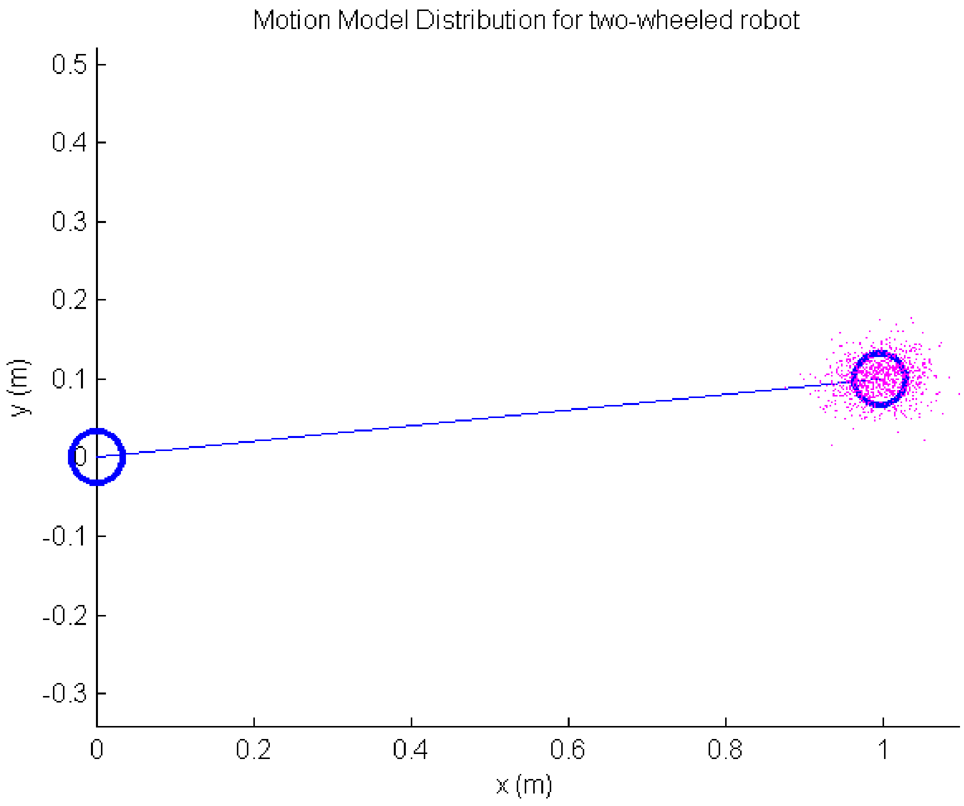
$$\varepsilon_t^{v} \sim N(0, R_t^{v})$$
$$\varepsilon_t^{\theta} \sim N(0, R_t^{\theta})$$

  - Motion Model

$$\begin{bmatrix} x_{1,t} \\ x_{2,t} \\ x_{3,t} \end{bmatrix} = g(x_{t-1}, u_t, \varepsilon_t) = \begin{bmatrix} x_{1,t-1} + \left(u_{1,t} + \varepsilon_t^{v}\right)\cos\left(x_{3,t-1} + \varepsilon_t^{\theta}\right)dt \\ x_{2,t-1} + \left(u_{1,t} + \varepsilon_t^{v}\right)\sin\left(x_{3,t-1} + \varepsilon_t^{\theta}\right)dt \\ \left(x_{3,t-1} + \varepsilon_t^{\theta}\right) + u_{2,t}dt \end{bmatrix}$$

Kinematic Model

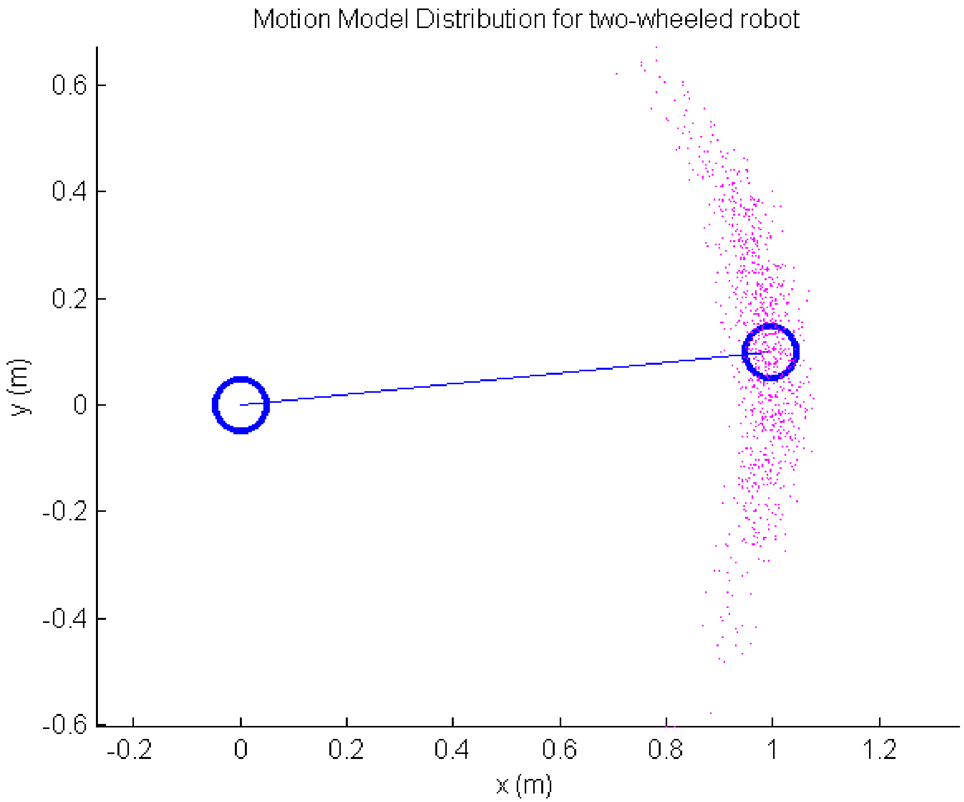
- Example: Two-wheeled robot with nonlinear disturbances (accurate steering and velocity control)



$$R = \begin{bmatrix} 0.001 & 0 \\ 0 & 0.0005 \end{bmatrix}$$

TWO-WHEELED KINEMATIC MODEL

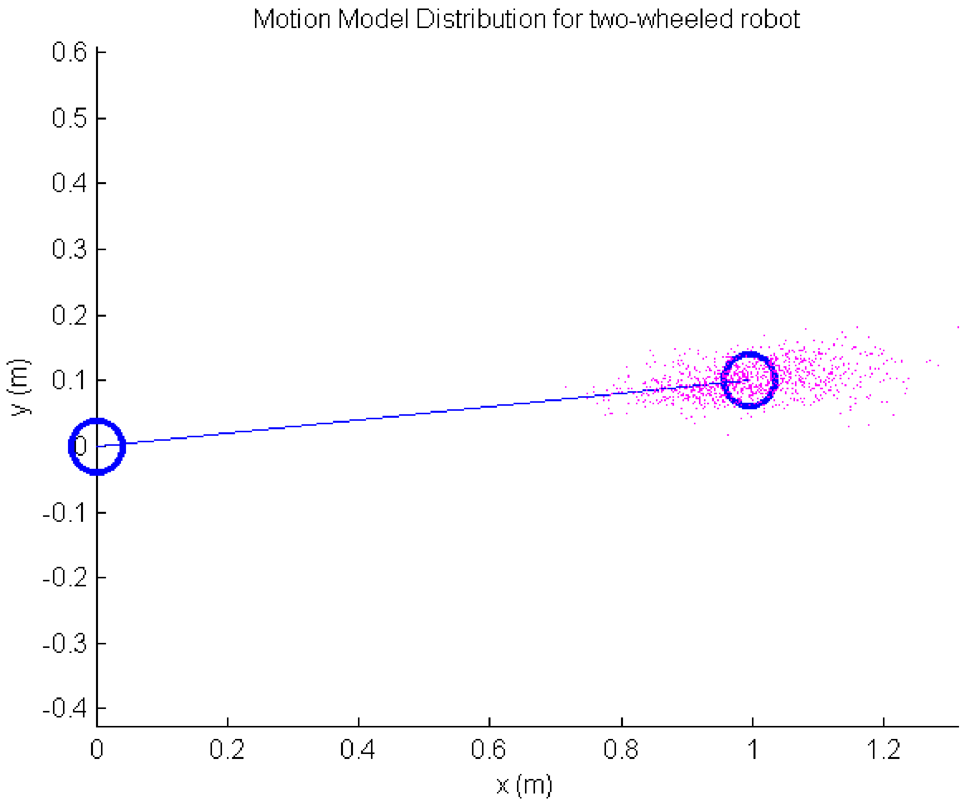
- Example: Two-wheeled robot with nonlinear disturbances (poor steering)



$$R = \begin{bmatrix} 0.001 & 0 \\ 0 & 0.05 \end{bmatrix}$$

TWO-WHEELED KINEMATIC MODEL

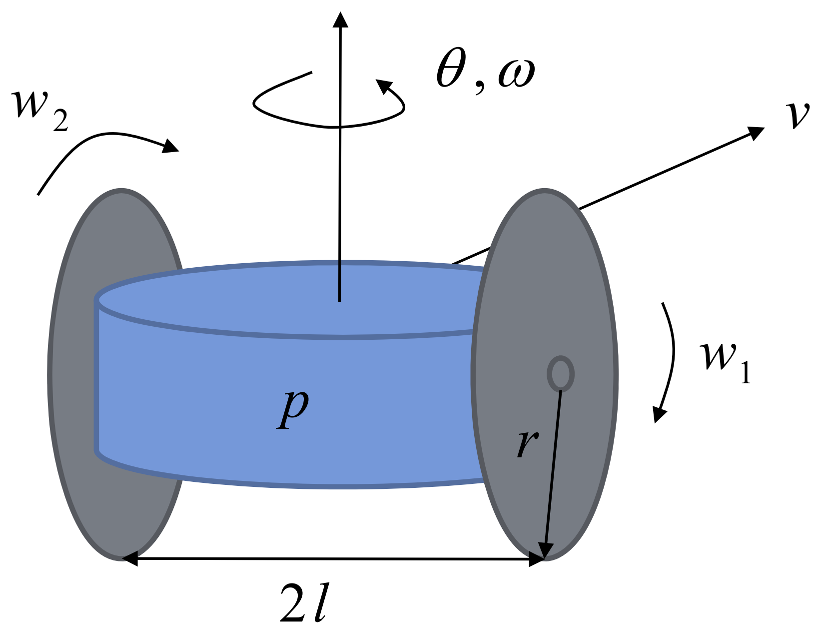
- Example: Two-wheeled robot with nonlinear disturbances (poor velocity control)



$$R = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.0005 \end{bmatrix}$$

Two-Wheeled Kinematic Model

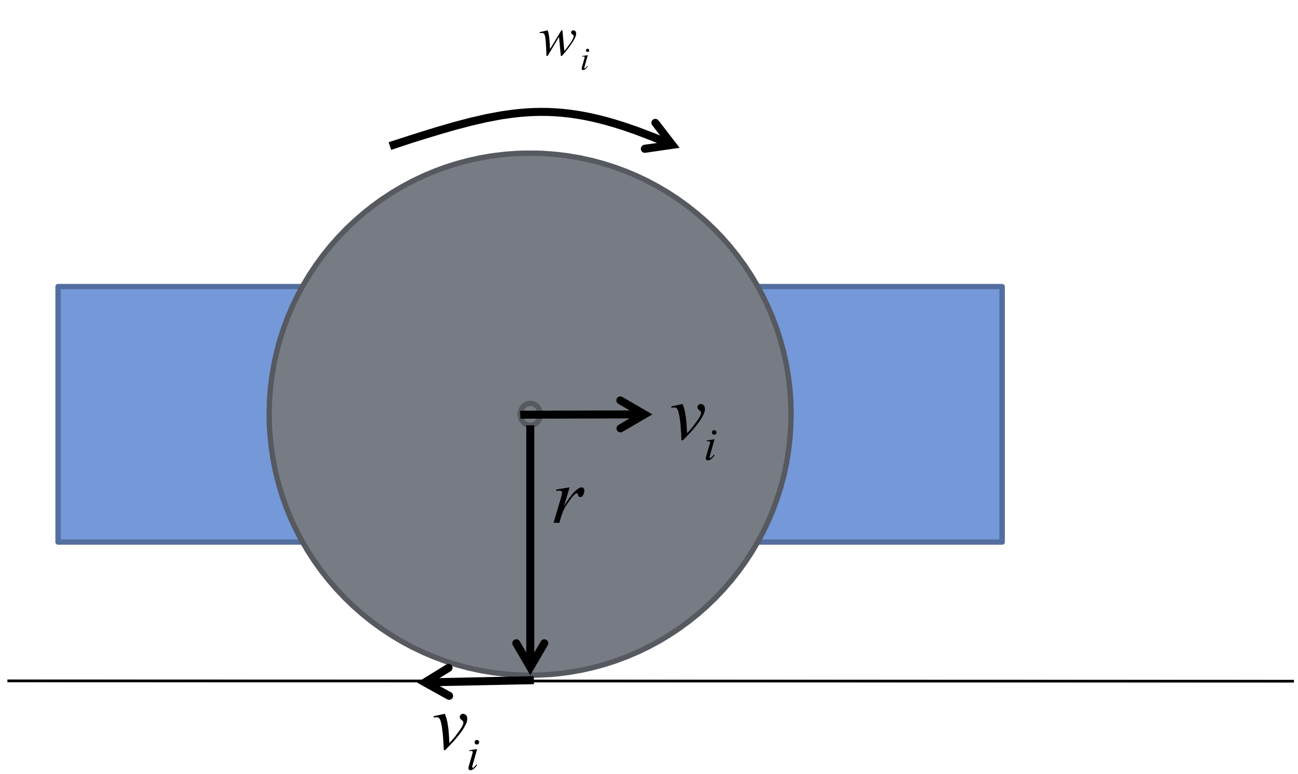
- If the control inputs are wheel speeds, can augment the model as follows:
  - Center: p
  - Wheel to center: l
  - Wheel radius: r
  - Wheel rotation rates: w_1, w_2



Two-Wheeled Kinematic Model

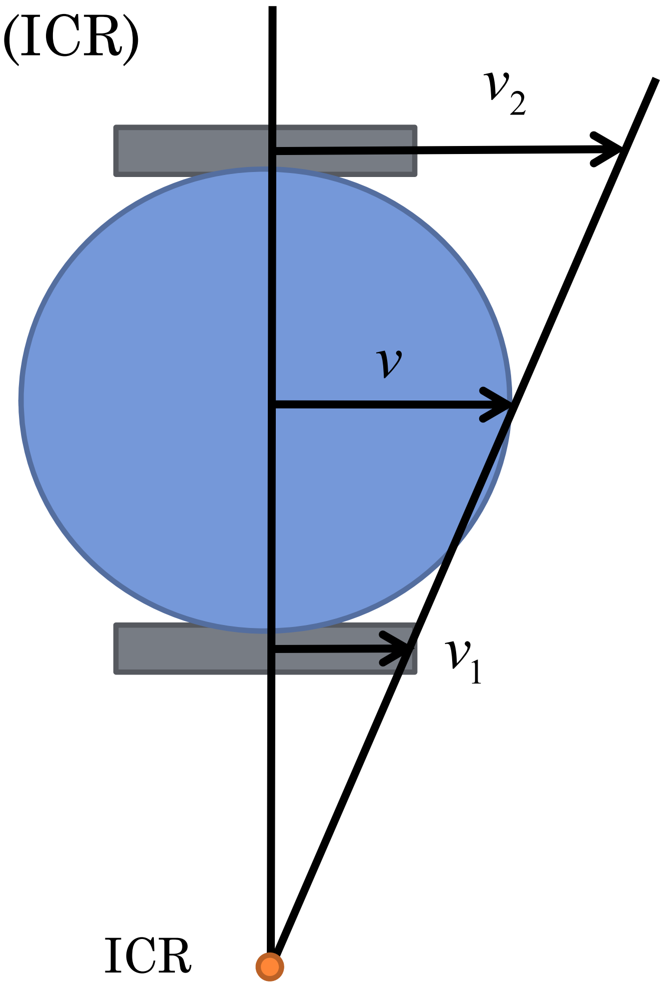
- Kinematic constraint

$$v_i = r w_i$$



TWO-WHEELED KINEMATIC MODEL

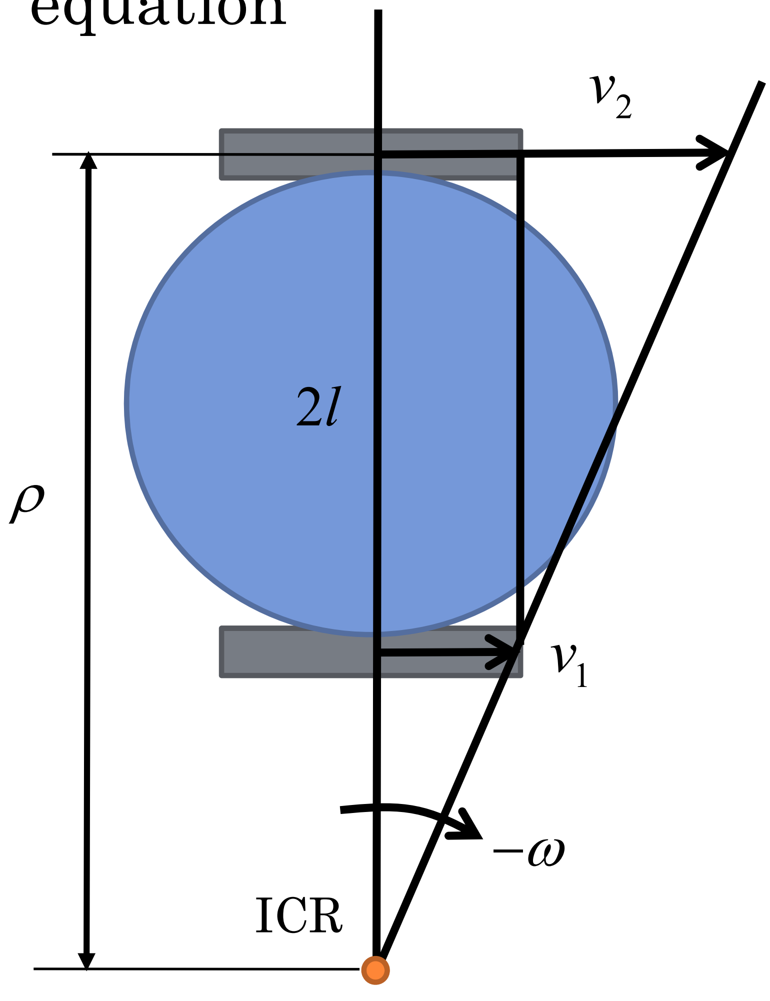
- Velocity is the average of the two wheel velocities
  - We can use the instantaneous centre of rotation (ICR)



$$v = \frac{v_1 + v_2}{2} = \frac{rw_1 + rw_2}{2}$$

Two-Wheeled Kinematic Model

- Equivalent triangles give the angular rate equation



$$\omega = \frac{-v_2}{\rho} = \frac{-(v_2 - v_1)}{2l}$$

$$\omega = \frac{rw_1 - rw_2}{2l}$$

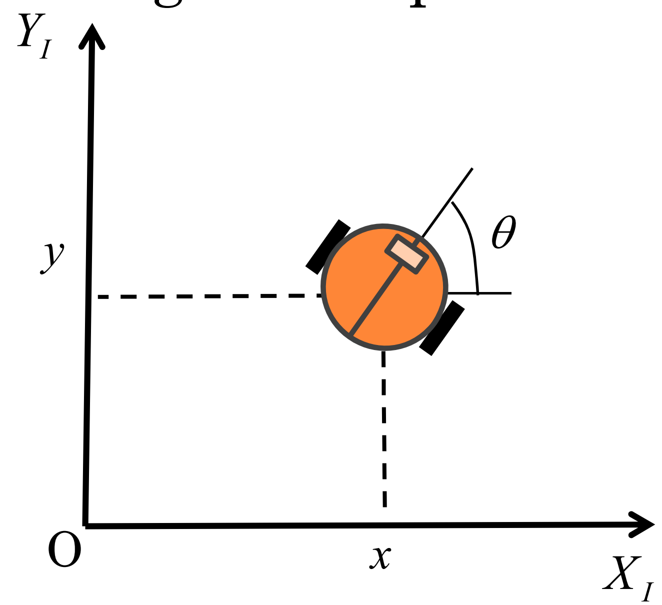
Two-Wheeled Kinematic Model

- So we now have a pair of equality constraints that relate the wheel rotation rates to the speed and rotation rate of the vehicle. Return to the standard 2 wheel robot, but change the inputs
  - Vehicle State:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x \\ y \\ \theta \end{bmatrix}$$

  - Inputs:

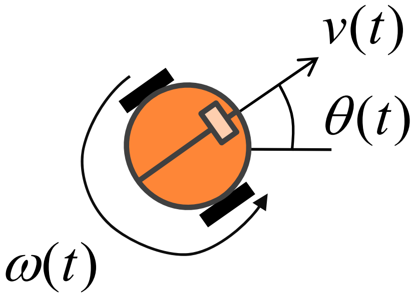
$$\begin{bmatrix} u_{t,1} \\ u_{t,2} \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$



Two-Wheeled Kinematic Model

- Summarizing the kinematic model in body coordinates

$$\begin{bmatrix} v \\ \omega \end{bmatrix} = \begin{bmatrix} \dfrac{rw_1 + rw_2}{2} \\ \dfrac{rw_1 - rw_2}{2l} \end{bmatrix}$$

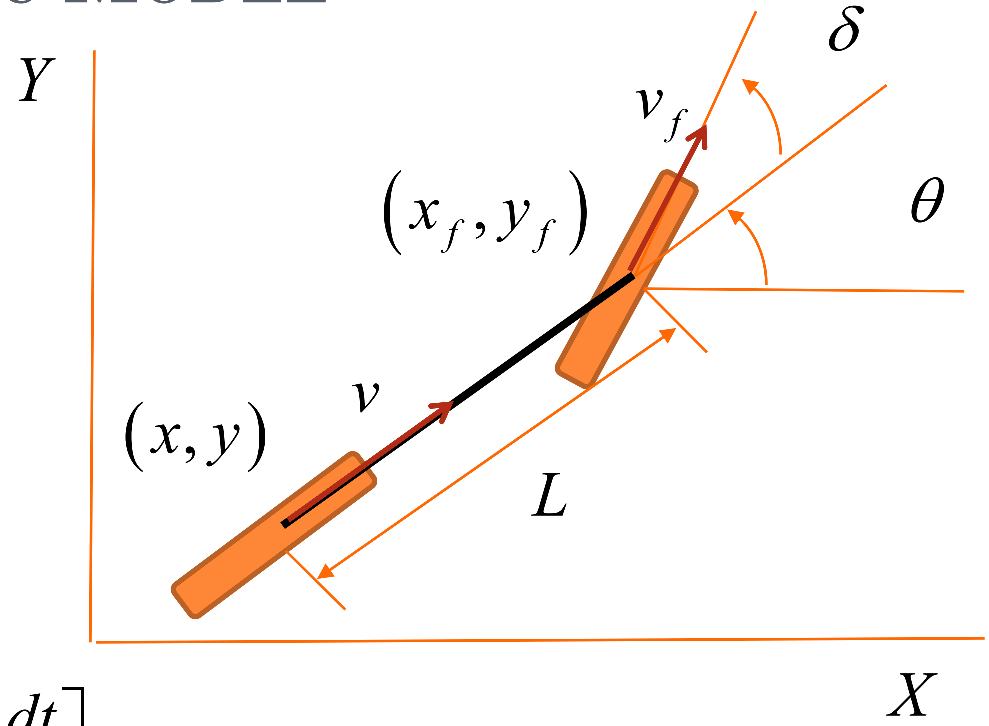


- Finally, the full dynamics of the vehicle are

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} v\cos\theta \\ v\sin\theta \\ \omega \end{bmatrix} = \begin{bmatrix} \dfrac{rw_1 + rw_2}{2}\cos\theta \\ \dfrac{rw_1 + rw_2}{2}\sin\theta \\ \dfrac{rw_1 - rw_2}{2l} \end{bmatrix}$$

Bicycle Kinematic Model

- Bicycle model
  - Front wheel steering
  - Track motion of rear wheel
  - Rear x, y dynamics same as before



$$\begin{bmatrix} x_t \\ y_t \end{bmatrix} = \begin{bmatrix} x_{t-1} + v_t \cos\theta_{t-1} dt \\ y_{t-1} + v_t \sin\theta_{t-1} dt \end{bmatrix}$$

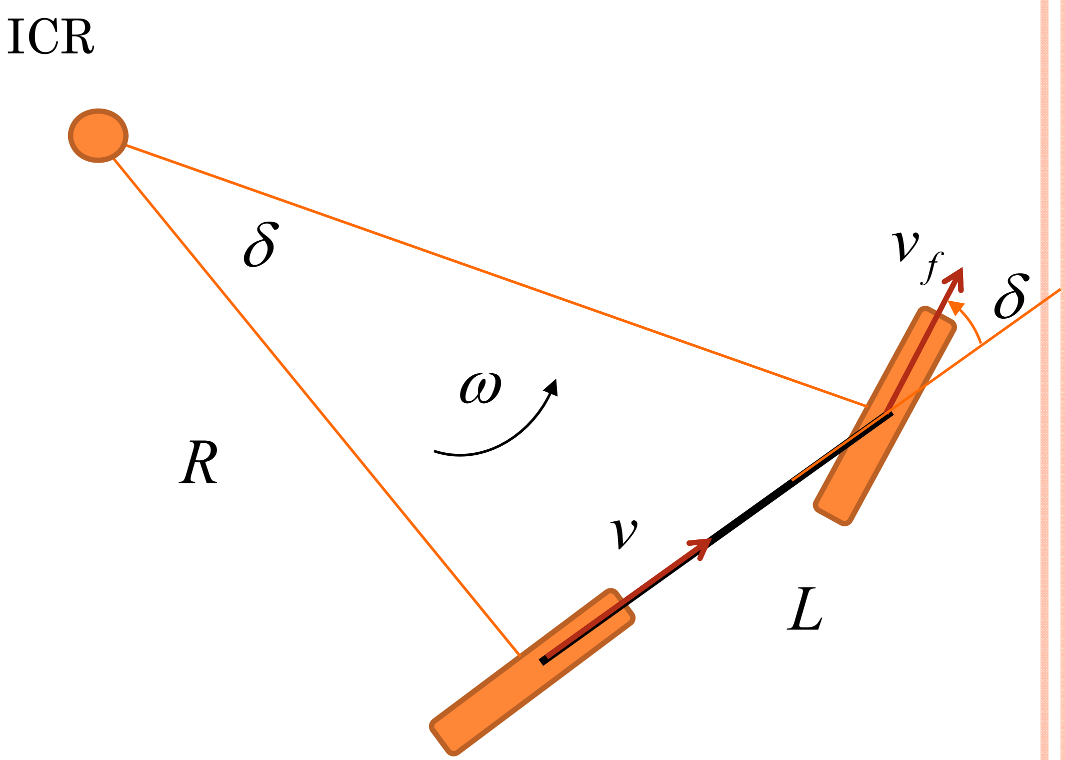
  - Front wheel $x_f = x + L\cos\theta, \quad y_f = y + L\sin\theta$

$$\begin{bmatrix} x_{f,t} \\ y_{f,t} \end{bmatrix} = \begin{bmatrix} x_{f,t-1} + v_t^f \cos(\theta_{t-1} + \delta_t) dt \\ y_{f,t-1} + v_t^f \sin(\theta_{t-1} + \delta_t) dt \end{bmatrix}$$

Bicycle Kinematic Model

- For rotation, we rely on the Instantaneous Center of Rotation (ICR) again

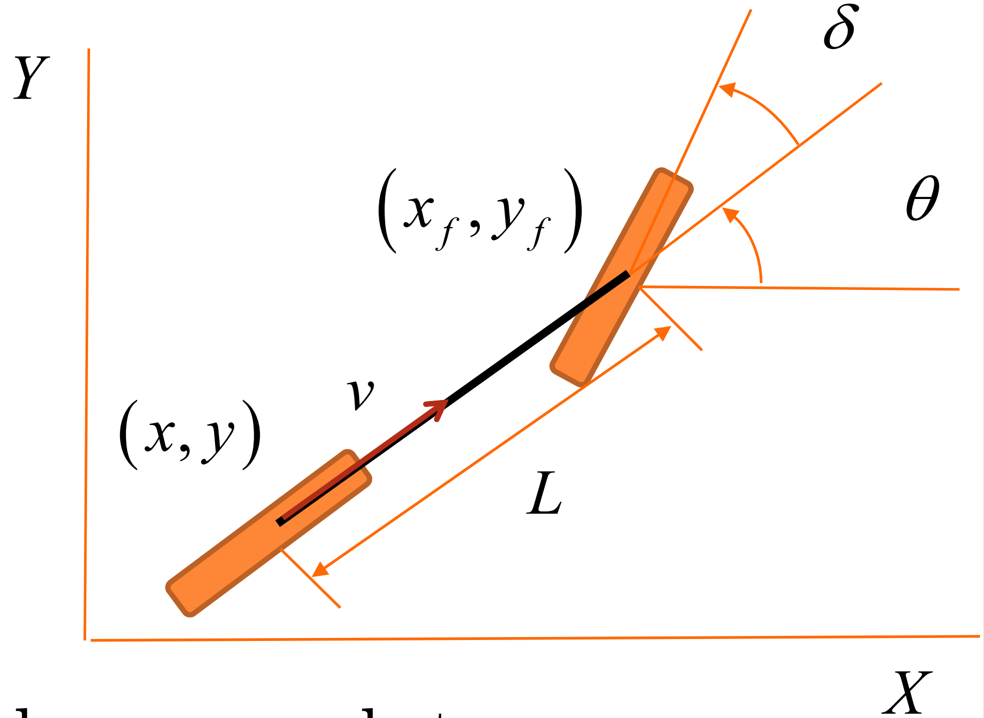
1. $$\omega = \frac{v}{R}$$
2. $$\tan\delta = \frac{L}{R}$$
3. $$\omega = \frac{v}{R} = \frac{v\tan\delta}{L}$$
4. $$\theta_t = \theta_{t-1} + \frac{v_t \tan\delta_t}{L} dt$$



Bicycle Kinematic Model

- Standard bicycle model

$$\begin{bmatrix} x_t \\ y_t \\ \theta_t \end{bmatrix} = \begin{bmatrix} x_{t-1} + v_t \cos\theta_{t-1} dt \\ y_{t-1} + v_t \sin\theta_{t-1} dt \\ \theta_{t-1} + \dfrac{v_t \tan\delta_t}{L} dt \end{bmatrix}$$



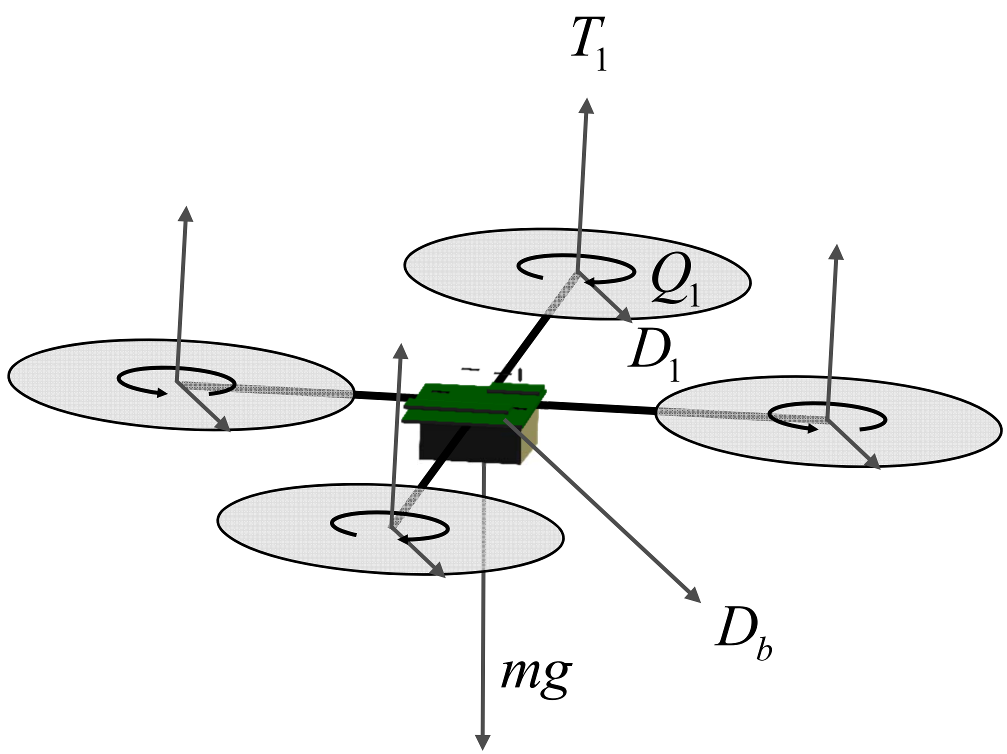
- Good model for cars, Ackermann robots
  - Add drivetrain dynamics which affect speed control

Dynamic Models

- For some robots, kinematics are insufficient to describe relationship between inputs and vehicle state
- Dynamics include forces and moments acting on robot in motion model
- Process:
  - Draw Free Body Diagram
  - Define equations of motion
  - Model forces and moments acting on vehicle

Dynamic Models

- Example: Quadrotor helicopter
- Free Body Diagram



Dynamic Models - Standard forms (Euler)

- Inertial frame Mixed frame Body frame

Inertial frame:

$$\dot{p}_I = v_I$$
$$m\dot{v}_I = F_I$$
$$\dot{\Theta} = \bar{R}_E \omega_B$$
$$\omega_I = R_B^I \omega_B$$
$$J_I \dot{\omega}_I = M_I$$

Mixed frame:

$$\dot{p}_I = v_I$$
$$m\dot{v}_I = F_I$$
$$\dot{\Theta} = \bar{R}_E \omega_B$$
$$J\dot{\omega}_B + \omega_B \times J\omega_B = M_B$$

Body frame:

$$\dot{p}_I = v_I$$
$$v_I = R_B^I v_B$$
$$m\dot{v}_B + \omega_B \times m v_B = F_B$$
$$\dot{\Theta} = \bar{R}_E \omega_B$$
$$J\dot{\omega}_B + \omega_B \times J\omega_B = M_B$$

- Common matrix form

$$\begin{bmatrix} mI & 0 \\ 0 & J \end{bmatrix} \begin{bmatrix} \dot{v}_b \\ \dot{\omega}_b \end{bmatrix} + \begin{bmatrix} \omega_b \times m v_b \\ \omega_b \times J\omega_b \end{bmatrix} = \begin{bmatrix} F_B \\ M_B \end{bmatrix}$$

Dynamic Models

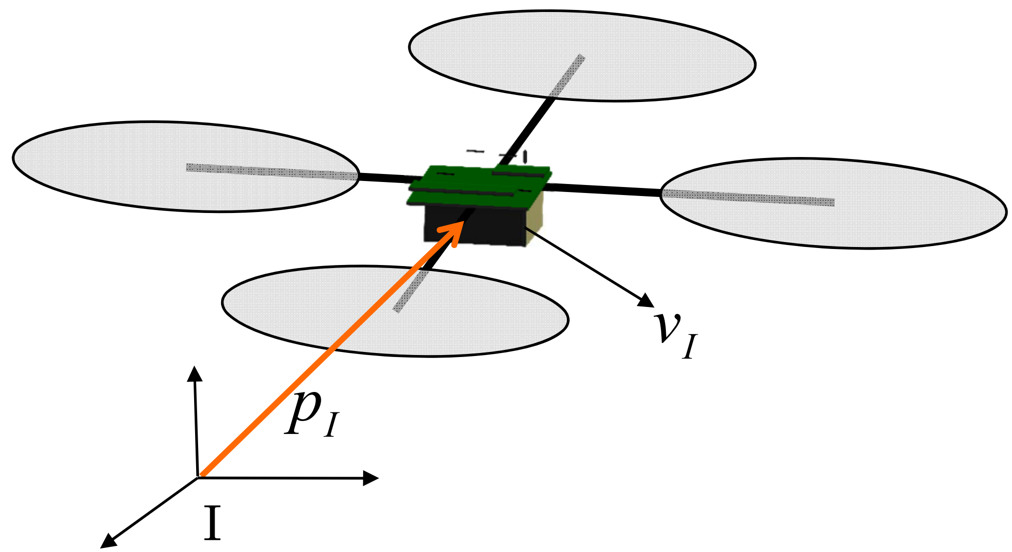
- Equations of motion
  - Standard 6 DOF motion model

$$\dot{p}_I = v_I$$

$$m\dot{v}_I = F_I$$

$$\dot{\Theta} = \bar{R}_E \omega_B$$

$$J\dot{\omega}_B = M_B - \omega_B \times J\omega_B$$



  - Can be used for any rigid body that translates and rotates in 3D
  - Naturally aligns with inertial (GPS) position and body (gyro) angular rate measurement

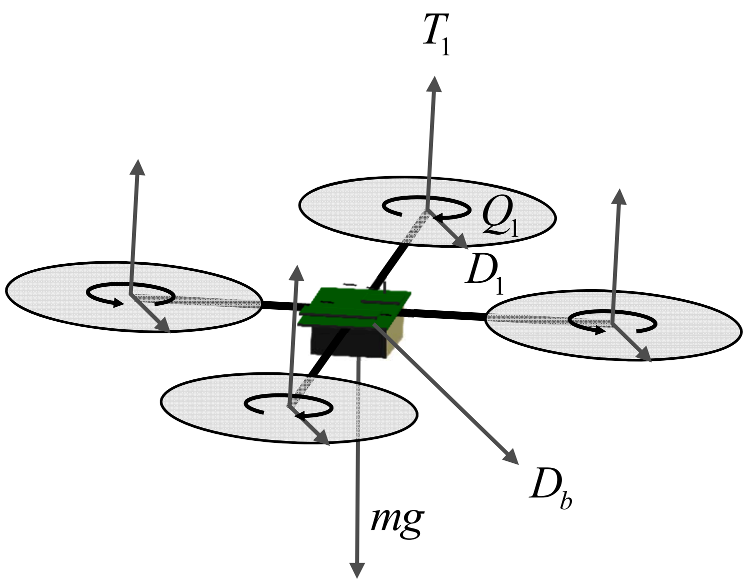
Dynamics

- Forces acting on vehicle

$$F_I = R_B^I \left(\sum_{i=1}^{4} (-T_i \hat{\boldsymbol{z}} + D_i \hat{\boldsymbol{e}}_h) \right) + D_B \hat{\boldsymbol{e}}_\infty + mg\hat{\mathbf{d}}$$

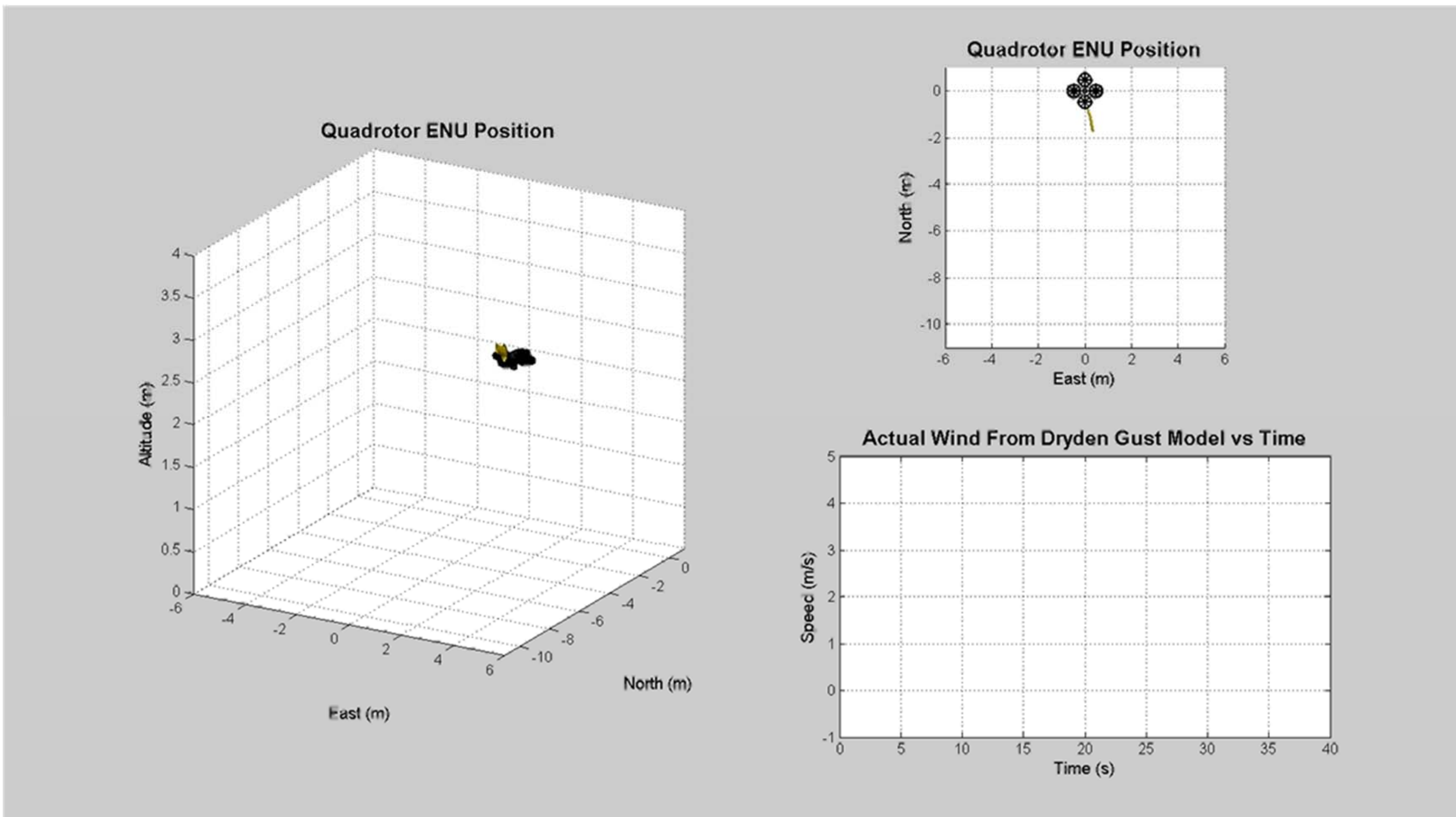
- Moments acting on vehicle

$$M_B = \sum_{i=1}^{4} \left((-1)^i Q_i \hat{\boldsymbol{z}} + T_i (\boldsymbol{r_i} \times \hat{\boldsymbol{z}}) + D_i (\boldsymbol{r_i} \times \hat{\boldsymbol{e}}_h) \right)$$



Resulting Motion

- Trajectory Control in Windy Conditions

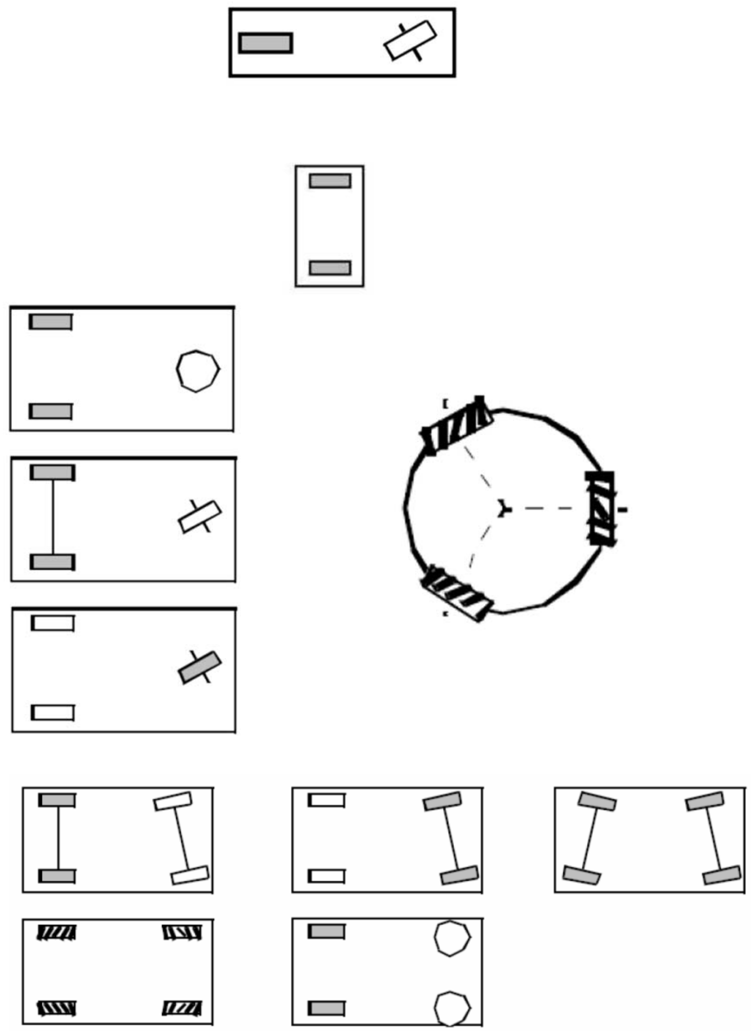


Carlos Wang, University of Waterloo, 2009

EXTRA SLIDES

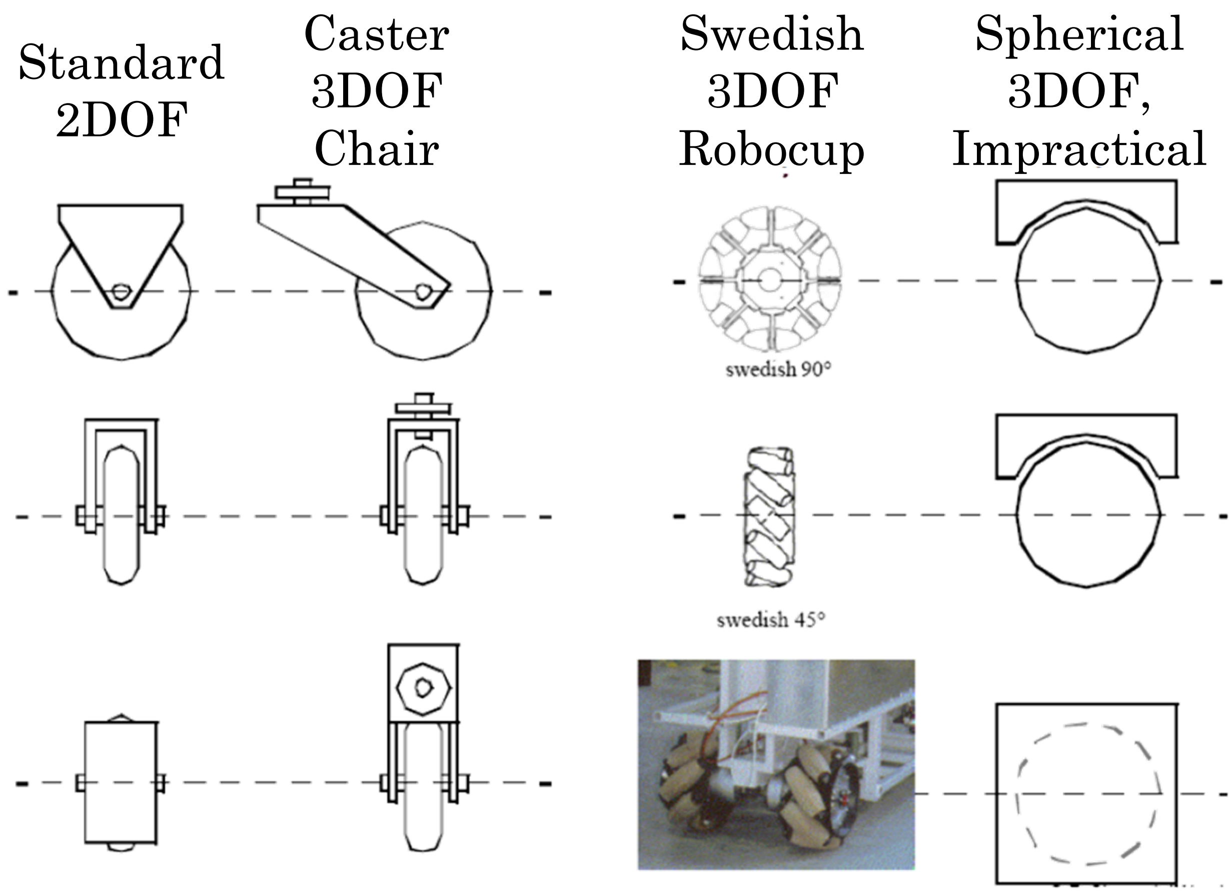
Wheel Configurations

- Two-wheeled
  - Bicycle
  - Segway
- Three-wheeled
  - Dolley
  - Tricycle
  - Big Wheel
  - Omni-directional
- Four-wheeled
  - Rear/Front/4 WD
  - Crazy vehicles



Wheeled Motion

- Kinematics governed by wheel number, type, geometry



WHEEL CHOICE

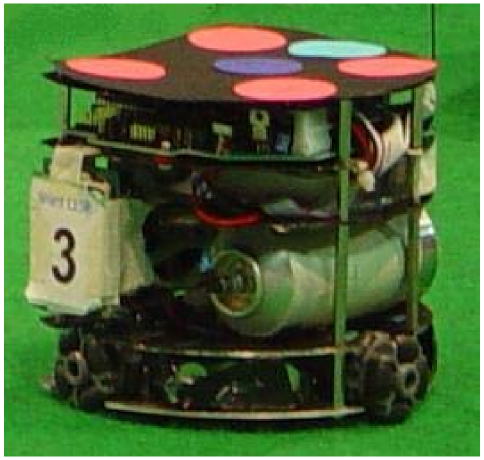


Image courtesy of Payam Sabzmeydani

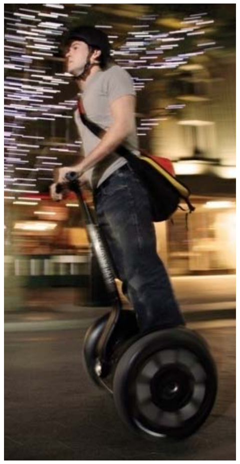


Image courtesy of Segway
Image courtesy of Segway

- Main Issues:
  - Stability
  - Maneuverability
  - Controllability
  - Mechanical Complexity

- Passive stability is guaranteed with 3 wheels, improved with 4 wheels
  - Active stability required with less (bicycle, Segway)

- Maneuverability/Controllability/Complexity
  - Combining steering and drive on one wheel difficult to realize but great for control (front wheel drive)

Uneven Terrain

- Suspension required to maintain contact, smooth sensor motion
- Bigger wheels effective for overcoming obstacles, but require more torque

Image courtesy of T. Barfoot

Image courtesy of EPFL

LEGGED MOTION

- Inspired by nature, less efficient, more maneuverable than wheeled motion
- Number of legs determines stability
- Number of joints determines maneuverability

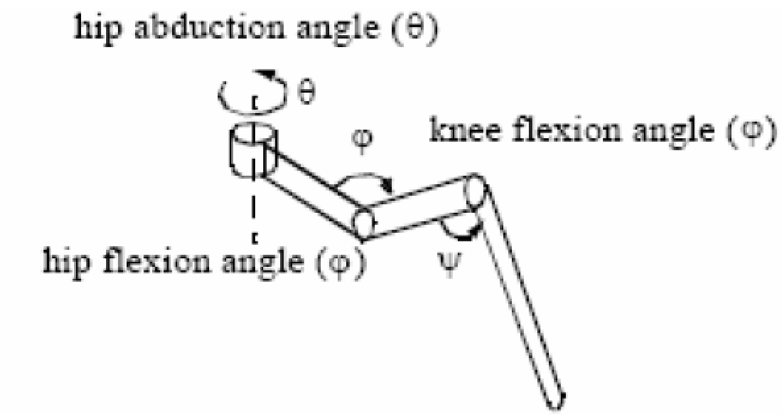
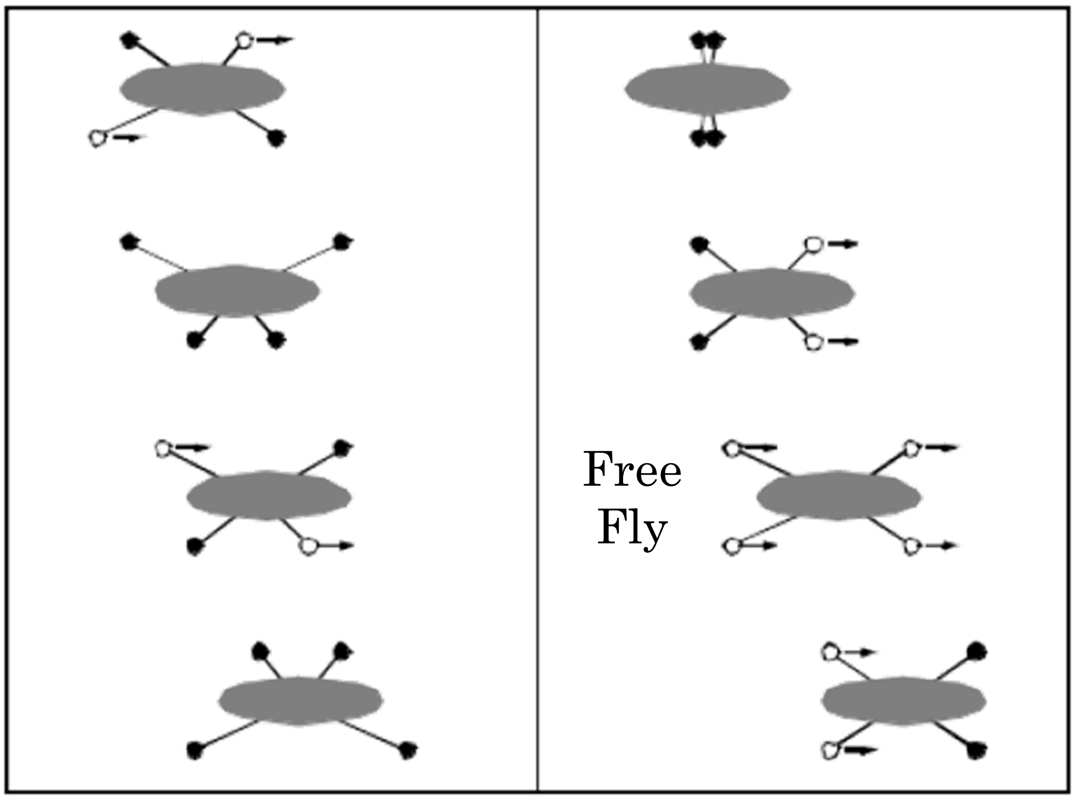


Image courtesy of Lego

LEGGED MOTION

- Usually requires predefined gait
  - Sequence of motions that achieves forward mobility
  - Dynamics quite complex, specialized



Changeover Walking

Galloping

LEGGED MOTION

- Once gait is defined, motion can be approximated by rolling polygon
  - Leg angle

$$\alpha = \sin^{-1}\left(\frac{d/2}{l}\right)$$

  - Polygon sides

$$n = \frac{\pi}{\alpha}$$

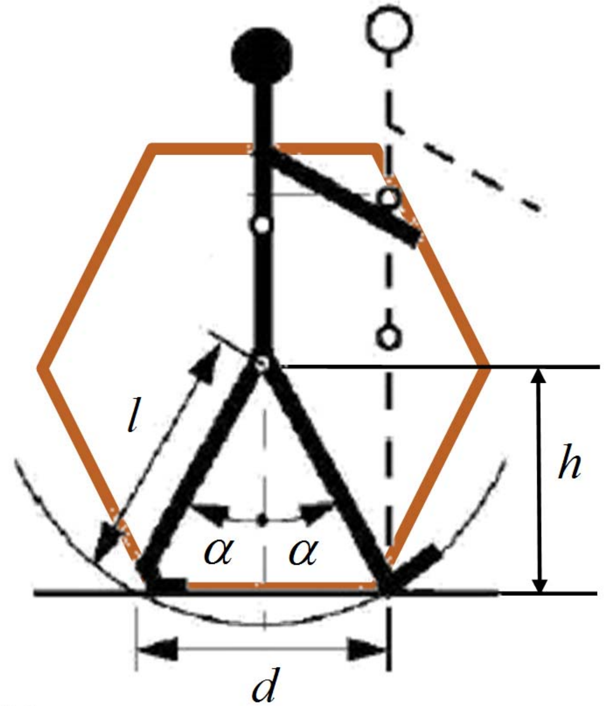
  - Hip height

$$h = l\cos\alpha$$

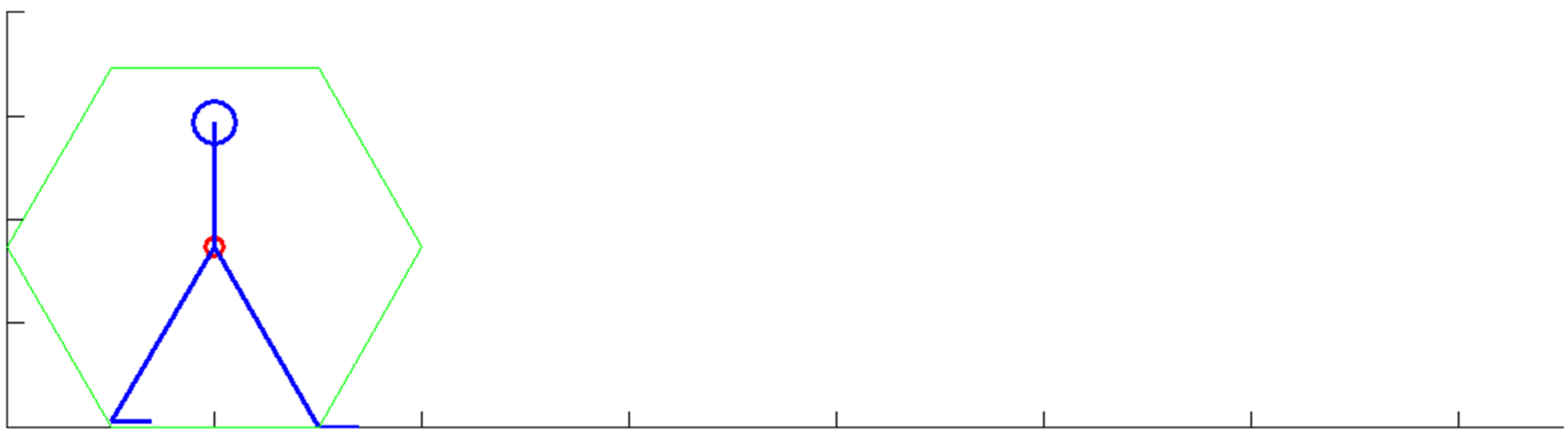
  - Forward speed

$$h = md\delta t$$

    - m steps per second, δt elapsed time



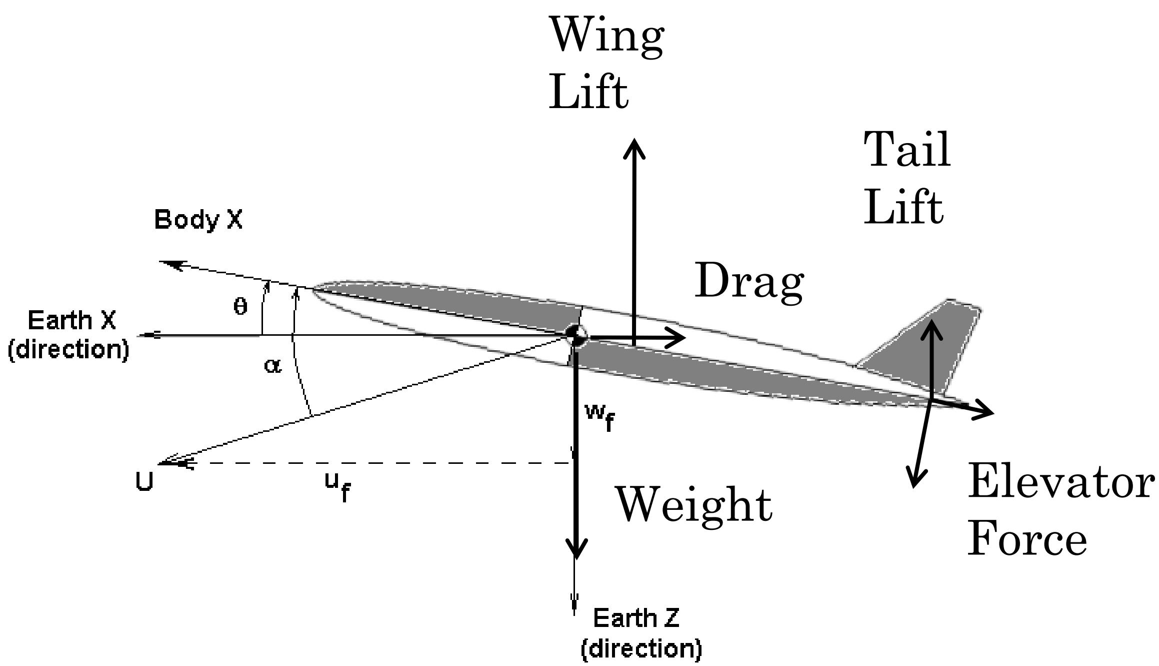
Legged Motion



- Red circle = hip height
- Leg length = step distance = 1
- Steps per second = 0.5

Aerial Motion

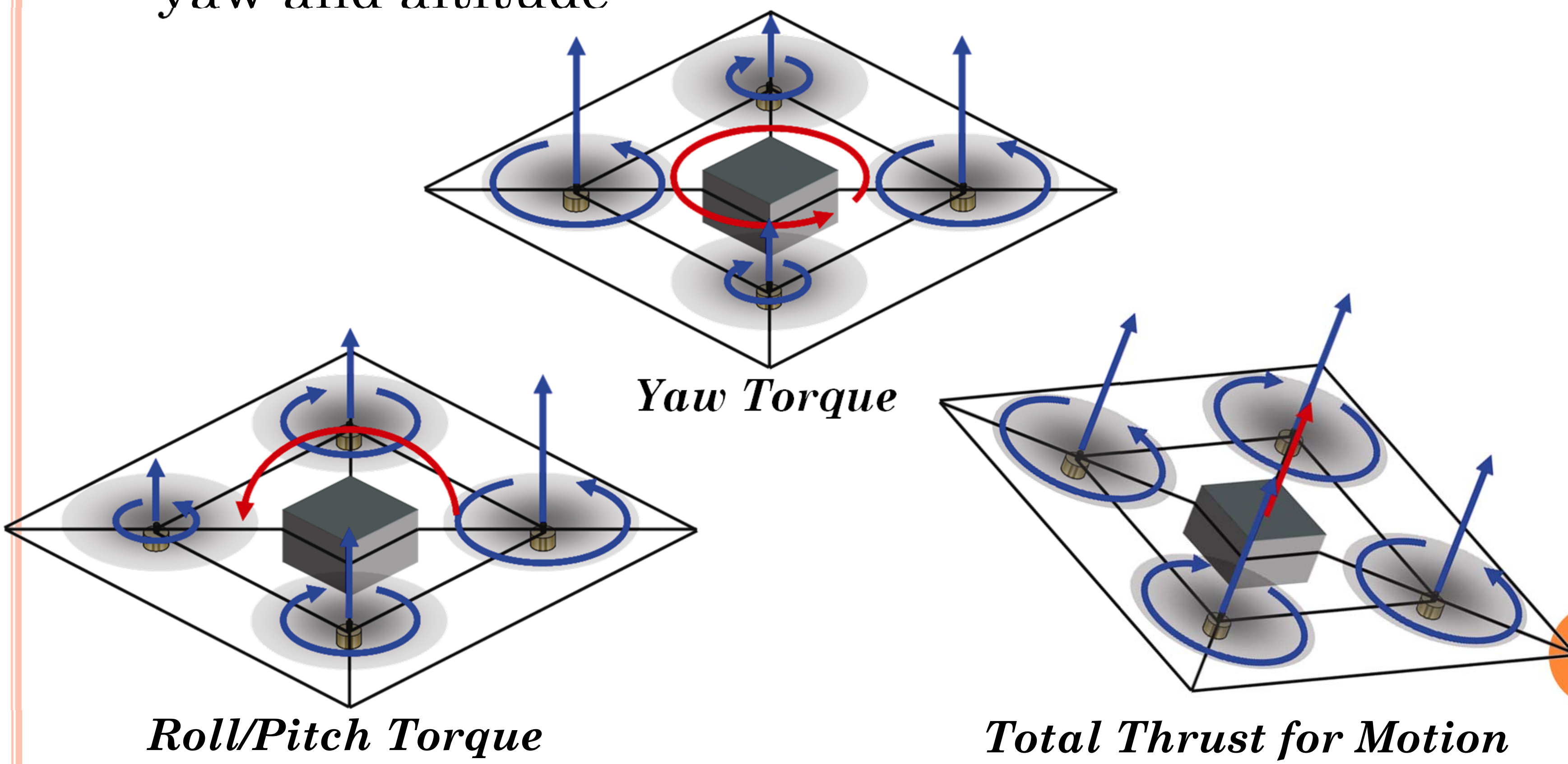
- Fixed Wing – Longitudinal Forces and Moments
  - Elevator causes moment about cg
  - Tail resists rotation about cg (damping)
  - Total lift and weight approximately balance
  - Drag increases with elevator deflection



Longitudinal Equations of Motion

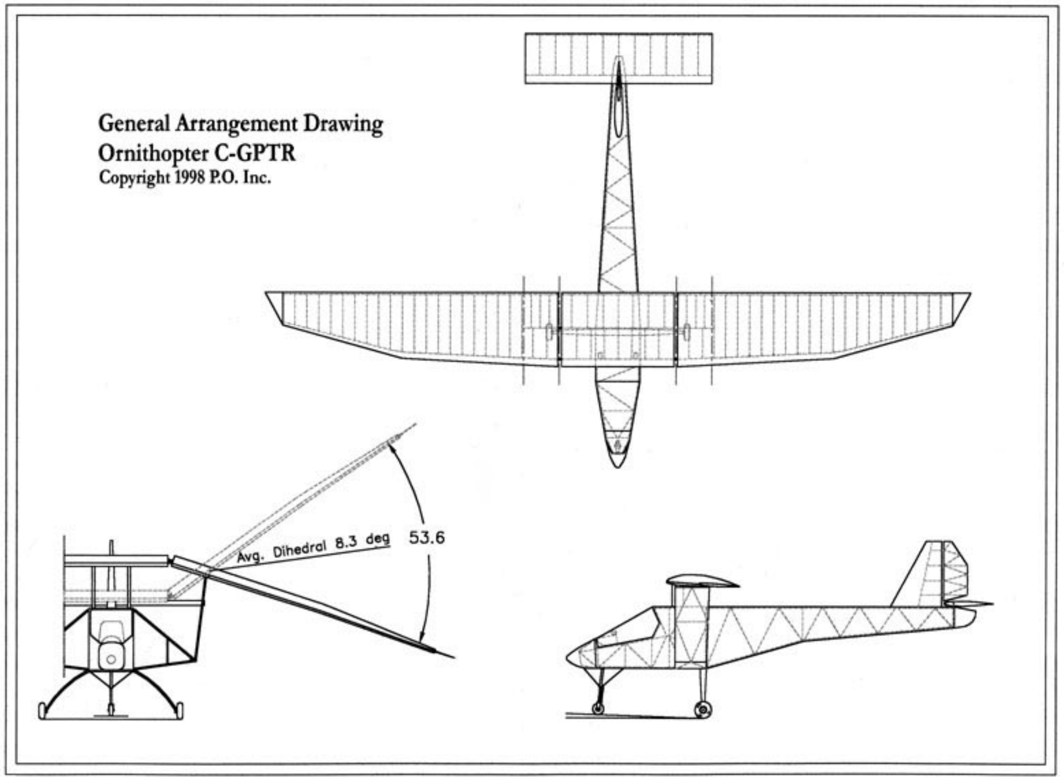
Aerial Motion

- Rotary wing: Quadrotors have two pairs of counter-rotating blades allow for fixed pitch rotors and independent actuation of roll, pitch, yaw and altitude



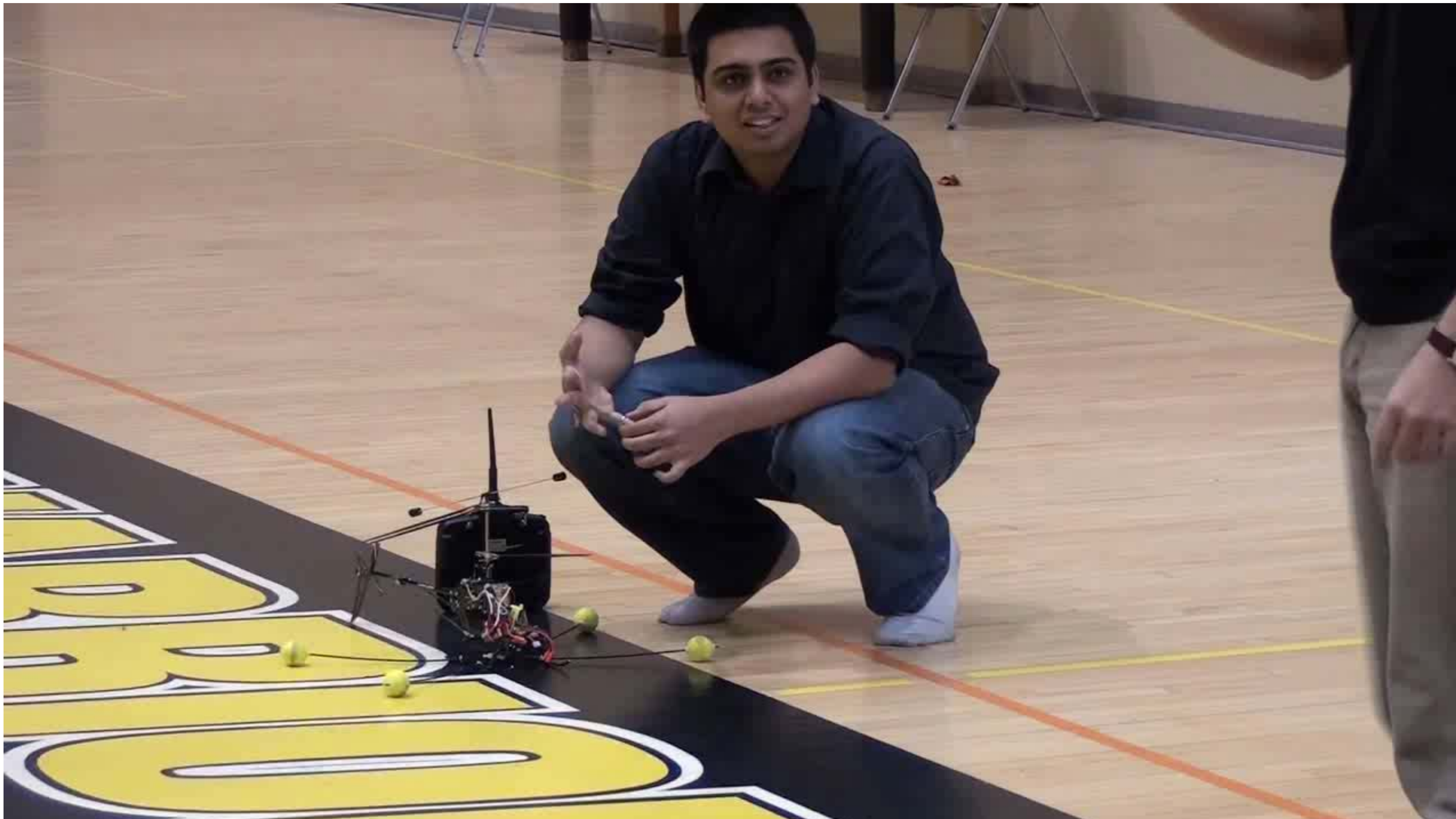
Aerial Motion

- Flapping Wing
  - Biologically inspired, difficult to achieve comparable efficiency to fixed, rotary wing
  - Similar approach to walking
    - Produce cyclic model of forces and moments



Dr. James DeLaurier, 2006
University of Toronto Aerospace Institute

UWMAV in Action



Aquatic Motion

- Ship motion, close to planar ground robot
  - Propulsion and steering at rear
  - Sideslip possible
  - Augmented model may include roll, pitch
- Submersible motion, analogous to fixed wing aircraft
  - Buoancy replaces lift to counteract gravity
- Swimming motion
  - Complex cyclic behaviour, can be modeled in a similar manner to walking, flapping

Aquatic Robots in Action

- World Autonomous Sailing Competition 2008

Swedish Wheel Robot